Lecture 1

1.1 Introduction

We will talk about n-variable calculus, that is, calculus on $\mathbb{R}^n$. Recall the following:

- The setting is,

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R}}_{n \text{ times}} = \{x = (x_1, \ldots, x_n) \mid x_i \in \mathbb{R} \, \forall \, i = 1, 2, \ldots, n\}$$

- Analysis on $\mathbb{R}$ consisted of ideas like open sets, compact sets, convergence, limits, differentiability, integrability etc.

- $\mathbb{R}^n$ is an n-dimensional inner product space over $\mathbb{R}$, with the standard orthonormal basis $\{e_j\}_{j=1}^n$

Extending the analytic ideas to $\mathbb{R}^n$ exploiting the algebraic structure is the matter of this course, which further gives way to differential geometry.

1.2 Review: $\mathbb{R}^n$ as a vector space

(i) The standard orthonormal basis of $\mathbb{R}^n$ is $\{e_i\}_{i=1}^n$.

(ii) For all $x \in \mathbb{R}^n$, there is a unique representation

$$x = \sum_{i=1}^{n} x_i e_i, \, x_i \in \mathbb{R}$$

Thus we identify x with the *coordinates* $(x_1, x_2, \ldots, x_n)$.

(iii) Euclidean inner product on $\mathbb{R}^n$:
For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we define

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

1.3 Linear Functions

In doing analysis on $\mathbb{R}$, the main motive was to study functions $f : \mathbb{R} \to \mathbb{R}$ and their properties, namely continuity, differentiability, integrability etc. We now wish to do the same for functions $f : \mathbb{R}^n \to \mathbb{R}^m$ for arbitrary natural numbers n, m.
Two easy examples of such functions are:

(i) Constant maps

$$f : \mathbb{R}^n \to \mathbb{R}^m$$
$$\forall\, x \in \mathbb{R}^n, f(x) = a,\; a \in \mathbb{R}^m$$

(ii) Linear maps

A function $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear if for all $\alpha \in \mathbb{R}, x, y \in \mathbb{R}^n$, $L(\alpha x + y) = \alpha L(x) + L(y)$.

It turns out that linear maps are useful in understanding most other '*nice*' functions, and so we now look at these in more detail.

Let L be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Consider the domain $\{\alpha x + y \mid \alpha \in \mathbb{R}\}$, that is, the line through y in the direction of x. The image under L is,

$$\{\alpha Lx + Ly \mid \alpha \in \mathbb{R}\}$$

which is the line through Ly in the direction of Lx. Hence, L maps lines to lines.

Exercise. Is the converse also true?

Matrix representation of a linear map

Let $L : \mathbb{R} \to \mathbb{R}$ be a linear map. Then,

$$L(x) = xL(1) \quad \forall\, x \in \mathbb{R}$$

Therefore,

$$\mathcal{L}(\mathbb{R}, \mathbb{R}) := \{\text{set of all linear maps from } \mathbb{R} \text{ to } \mathbb{R}\} \leftrightarrow \mathbb{R}$$

Now consider the general case; let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. If we fix the bases $\{e_j\}_{j=1}^n$ of $\mathbb{R}^n$ and $\{e_i\}_{i=1}^m$ of $\mathbb{R}^m$, L is determined uniquely by the equations

$$Le_j = \sum_{i=1}^{m} a_{ij} e_i$$

and hence,

$$L \leftrightarrow (a_{ij})_{m \times n} \in M_{m,n}(\mathbb{R})$$

1.4 Analytic ideas in $\mathbb{R}^n$

We have the Euclidean norm on $\mathbb{R}^n$ defined by,

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \qquad \forall\, x \in \mathbb{R}^n$$

This induces the metric given as,

$$d(x, y) = \|x - y\| = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}} \qquad \forall\, x, y \in \mathbb{R}^n$$

Theorem 1.4.1 (Cauchy-Schwarz Inequality)

For all $x, y \in \mathbb{R}^n$,

$$\langle x, y \rangle \le \|x\|\|y\|$$

Proof. Consider $x, y \in \mathbb{R}^n$. We have,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i y_j - x_j y_i)^2 \geq 0$$

But the left-hand side is, after expanding,

$$\sum_{i,j=1}^{n} x_i^2 y_j^2 + \sum_{i,j=1}^{n} x_j^2 y_i^2 - 2\sum_{i,j=1}^{n} x_i x_j y_i y_j = 2\|x\|^2\|y\|^2 - 2\langle x, y\rangle^2$$

which gives the desired inequality. □

Note: The proof shows that equality holds only if there is $\lambda \in \mathbb{R}$ such that for all i, either $x_i = \lambda y_i$ or $y_i = \lambda x_i$.

Recall the triangle inequality for $\mathbb{R}$, for all $x, y \in \mathbb{R}$

$$|x + y| \leq |x| + |y|$$

Theorem 1.4.2 (Triangle inequality for $\mathbb{R}^n$)

For all $x, y \in \mathbb{R}^n$,

$$\|x + y\| \leq \|x\| + \|y\|$$

Proof. We have,

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \\
&\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| && \text{(Cauchy Schwarz inequality)} \\
\implies \|x + y\| &\leq \|x\| + \|y\|
\end{aligned}$$

which is the desired inequality. □

The following is a technical result, which can be thought of as an analogue of the Lipschitz condition for linear maps on $\mathbb{R}^n$. It hints towards continuity of linear maps, and we will see that it is indeed so later.

Theorem 1.4.3

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. There is $M > 0$ such that

$$\|Lx\| \leq M\|x\| \quad \forall x \in \mathbb{R}^n$$

Proof. We have, for $x = \sum_{i=1}^{n} x_i e_i$,

$$\begin{aligned}
\|Lx\| &= \left\|\sum_{i=1}^{n} x_i L e_i\right\| \\
&\leq \sum_{i=1}^{n} |x_i|\|Le_i\| && \text{(Triangle inequality)} \\
\implies \|Lx\| &\leq \|x\|\left(\sum_{i=1}^{n}\|Le_i\|^2\right)^{\frac{1}{2}} && \text{(Cauchy Schwarz inequality)}
\end{aligned}$$

Taking $M = \left(\sum_{i=1}^{n}\|Le_i\|^2\right)^{\frac{1}{2}}$, we get the result. □