Lecture 2

2.1 Distance and Topology in $\mathbb{R}^n$

Using the inner product on $\mathbb{R}^n$, we get the Euclidean distance

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle} = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

We wish to extend the notions of $(\mathbb{R}, |\cdot|)$ to $(\mathbb{R}^n, \|\cdot\|)$. We already know from Lecture 1 that the triangle inequality holds,

$$\|x+y\| \leq \|x\| + \|y\|$$

Definition 2.1.1 ▶ Open balls

The open ball centered at $a \in \mathbb{R}^n$ of radius r is,

$$B_r(a) = \{x \in \mathbb{R}^n \mid \|x-a\| < r\}$$

Exercise. Show that open balls are convex sets.

Definition 2.1.2 ▶ Open sets

A set $\mathcal{O} \subseteq \mathbb{R}^n$ is open if $\mathcal{O} = \phi$ or for all $x \in \mathcal{O}$, there is $r > 0$ such that $B_r(x) \subseteq \mathcal{O}$.

Example 2.1.1

(i) Any open ball is open.

(ii) We define open boxes in $\mathbb{R}^n$ to be the subsets of the form $\prod_{i=1}^{n}(a_i, b_i)$. Any open box is open.

Definition 2.1.3 ▶ Convergence of Sequences

Let $\{x_m\}_{m\in\mathbb{N}} \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. We say $x_m \longrightarrow x$ if for all $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\begin{aligned} &\|x_m - x\| < \varepsilon \,\forall\, m \geq N \\ \iff\ & d(x_m, x) < \varepsilon \quad \forall\, m \geq N \\ \iff\ & x_m \in B_\varepsilon(x) \quad \forall\, m \geq N \end{aligned}$$

Exercise. Show that the limit of a sequence in $\mathbb{R}^n$ is unique whenever it exists.

Definition 2.1.4 ▶ Limit points

We define the deleted ε-neighbourhood of $a \in \mathbb{R}^n$ to be $D_\varepsilon(a) = B_\varepsilon(a) \setminus \{a\}$. The point a is a limit point of $S \subseteq \mathbb{R}^n$ if for all $\varepsilon > 0, D_\varepsilon(a) \cap S \neq \phi$. If we do not delete a, we get isolated points.

Definition 2.1.5 ▶ Projections

For all $i \in \{1, 2, \ldots, n\}$ we define the maps

$$\begin{aligned} \Pi_i : \mathbb{R}^n &\to \mathbb{R} \\ x = (x_1, \ldots, x_n) &\mapsto x_i \end{aligned}$$

Π_i is called the projection onto the ith coordinate.

Theorem 2.1.1

Let $\{x_m\}_{m\in\mathbb{N}} \cup \{x\} \subseteq \mathbb{R}^n$. Then,

$$\begin{aligned} & x_m \longrightarrow x \\ \iff\ & \Pi_i(x_m) \longrightarrow \Pi_i(x) \,\forall\, i \in \{1, 2, \ldots, n\} \end{aligned}$$

Proof. Assume $x_m \longrightarrow x$. Now, for all $j \in \{1, 2, \ldots, n\}$,

$$\begin{aligned} \|x_m - x\|^2 = \sum_{i=1}^{n} |\Pi_i(x_m) - \Pi_i(x)|^2 &\geq |\Pi_j(x_m) - \Pi_j(x)|^2 \\ \implies |\Pi_j(x_m) - \Pi_j(x)| &\longrightarrow 0 \\ \implies \Pi_j(x_m) &\longrightarrow \Pi_j(x) \end{aligned}$$

Now assume $\Pi_j(x_m) \longrightarrow \Pi_j(x)$ for all $j \in \{1, 2, \ldots, n\}$. Then,

$$\begin{aligned} |\Pi_j(x_m) - \Pi_j(x)| &\longrightarrow 0, \,\forall\, j \in \{1, 2, \ldots, n\} \\ \implies \sum_{i=1}^{n} |\Pi_j(x_m) - \Pi_j(x)|^2 &\longrightarrow 0 \\ \implies \|x_m - x\|^2 &\longrightarrow 0 \\ \implies x_m &\longrightarrow x \end{aligned}$$

□

Definition 2.1.6 ▶ Closed sets

A set $C \subseteq \mathbb{R}^n$ is closed if $\mathbb{R}^n \setminus C$ is open.

Exercise. Show that a set $C \subseteq \mathbb{R}^n$ is closed iff $\forall\, \{x_m\}_{m\in\mathbb{N}} \subseteq C$ with $x_m \longrightarrow x$ for some $x \in \mathbb{R}^n$, we have $x \in C$.

Exercise. Show that:

(1) Arbitrary union of open sets is open.

(2) Finite intersection of open sets is open.

(3) Arbitrary intersection of closed sets is closed.

(4) Finite union of closed sets is closed.

(5) Any finite subset of $\mathbb{R}^n$ is closed.

Definition 2.1.7 ▶ Interior of a set

Let $\phi \neq S \subseteq \mathbb{R}^n$. The interior of S is,

$$\text{Int}(S) = \{a \in S \mid \exists\, r > 0, B_r(a) \subseteq S\}$$

Exercise. Show that:

(1) For any nonempty set $S \subseteq \mathbb{R}^n$, $\text{Int}(S)$ is open.

(2) A set S is open iff $\text{Int}(S) = S$.

Definition 2.1.8 ▶ Exterior of a set

Let $\phi \neq S \subseteq \mathbb{R}^n$. The exterior of S is,

$$\text{Ext}(S) = \{a \in \mathbb{R}^n \mid \exists\, r > 0, B_r(a) \cap S = \phi\}$$

Exercise. Show that $\text{Ext}(S) = \text{Int}(\mathbb{R}^n \setminus S)$.

Example 2.1.2

For $S = [0, 2] \setminus \{1\} = [0, 1) \cup (1, 2]$, $1 \notin \text{Ext}(S)$.

Definition 2.1.9 ▶ Boundary of a set

Let $\phi \neq S \subseteq \mathbb{R}^n$. The boundary of S is,

$$\partial S = \{a \in \mathbb{R}^n \mid \forall\, r > 0, B_r(a) \cap S \neq \phi \text{ and } B_r(a) \cap (\mathbb{R}^n \setminus S) \neq \phi\}$$

Example 2.1.3

For $S = [0, 1) \cup (1, 2] \cup \{5\}$, $\partial S = \{0, 1, 2, 5\}$ but the set of limit points is $\{0, 1, 2\}$.

Exercise. Show that:

(1) S is open iff $S \cap \partial S = \phi$.

(2) S is closed iff $S \supseteq \partial S$.

(3) S is closed iff $S = \overline{S} =: S \cup \partial S = S \cup \{\text{Limit points of } S\}$

(4) $\overline{S} = \text{Int}(S) \sqcup \partial S$. This gives the partition $\mathbb{R}^n = \text{Int}(S) \sqcup \partial S \sqcup \text{Ext}(S)$

(5) ∂S is closed.

(6) Let $\{\mathcal{O}_i\}_{i=1}^n \subseteq \mathcal{P}(\mathbb{R})$ and define $\mathcal{O} = \prod_{i=1}^n \mathcal{O}_i$. If $\mathcal{O}_i$'s are open (closed), $\mathcal{O}$ is open (closed).

2.2 Limits and Continuity

Recall the notion of limit in $\mathbb{R}$:
Suppose $f : (a,b) \setminus \{c\} \to \mathbb{R}$ is a function. We say $\lim_{x\to c} f$ exists if there is $\alpha \in \mathbb{R}$ such that $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $x \in D_\delta(c) \implies |f(x) - \alpha| < \varepsilon$, that is, $f(D_\delta(c)) \subseteq B_\varepsilon(\alpha)$; in such a case we say that $\lim_{x\to c} f = \alpha$.
We now extend this to $\mathbb{R}^n$.

Definition 2.2.1 ▶ Limits in $\mathbb{R}^n$

Let $S \subseteq \mathbb{R}^n, a \in \{\text{Limit points of } S\}$ and, $f : S \setminus \{a\} \to \mathbb{R}^m$. We say that $\lim_{x\to a} f = b$ if for all $\varepsilon > 0$, there is $\delta > 0$ such that $f(x) \in B_\varepsilon(b)$ for all $x \in D_\delta(a) \cap S$.
In other words, $\lim_{x\to a} f = b$ if for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|f(x) - b\| < \varepsilon \quad \forall\, x \in S, 0 < \|x - a\| < \delta$$

Definition 2.2.2 ▶ Continuity in $\mathbb{R}^n$

Let $S \subseteq \mathbb{R}^n, a \in S$ and, $f : S \to \mathbb{R}^m$. We say that f is continuous at a if for all $\varepsilon > 0$, there is $\delta > 0$ such that $f(x) \in B_\varepsilon(f(a))$ for all $x \in B_\delta(a) \cap S$, that is, $\|f(x) - f(a)\| < \varepsilon$ for all $x \in S$ with $\|x - a\| < \delta$.

Note: Any function defined on S is vacuously continuous at an isolated point a by our definition.