Lecture 3

3.1 Introduction

We denote the set of limit points of $S \subseteq \mathbb{R}^n$ by S'. Let $f : S \to \mathbb{R}^m$ and $a \in S$. f is continuous at a iff for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|f(x) - f(a)\| < \varepsilon \quad \forall \|x - a\| < \delta, x \in S$$
$$\iff$$
$$f(B_\delta(a) \cap S) \subseteq B_\varepsilon(f(a))$$

For $a \in S'$, we then get that f is continuous at a iff

$$\lim_{\|x-a\|\to 0} \|f(x) - f(a)\| = 0 \iff \lim_{\|h\|\to 0} \|f(a+h) - f(a)\| = 0$$

But $\|h\| \to 0 \iff h \to 0$, and so we have f is continuous at $a \in S'$ iff

$$\lim_{h\to 0} \|f(a+h) - f(a)\| = 0$$

The proof of the following theorem is left as an exercise.

Theorem 3.1.1

Let $S \subseteq \mathbb{R}^n, a \in S', b \in \mathbb{R}^m, f : S \to \mathbb{R}^m$. The following are equivalent:

(i) $\lim_{x\to a} f = b$

(ii) $\forall \{x_p\} \subseteq S \setminus \{a\}$ with $x_p \longrightarrow a$, we have $f(x_p) \longrightarrow b$

(iii) $\lim_{x\to a} \|f(x) - b\| = 0$

Note: If $a \in S$, we can take $b = f(a)$ and get analogous results for continuity at a.

3.2 Properties of Continuous functions

Definition 3.2.1 ▶ Continuity on sets

Let $S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}^m$. f is continuous on S if it is continuous at all $a \in S$.

Note: It is convenient to take S to be open, as f is continuous at any isolated points of S vacuously.

Theorem 3.2.1

Let $S \subseteq \mathbb{R}^n, f : S \to \mathbb{R}^m$. The following are equivalent:

(1) f is continuous on S.

(2) $\forall \{x_p\} \subseteq S$ with $x_p \longrightarrow a \in S$, we have $f(x_p) \longrightarrow f(a)$

(3) (Assuming S is open) $f^{-1}(\mathcal{O})$ is open for all $\mathcal{O} \subseteq \mathbb{R}^m$ open.

(4) (Assuming S is open) $f^{-1}(C)$ is closed for all $C \subseteq \mathbb{R}^m$ closed.

Proof. We have the following cases.

- (3) $\iff$ (4)
  This is true as if $g : X \to Y$ then for all $A \subseteq Y$, $g^{-1}(Y \setminus A) = X \setminus g^{-1}(A)$.

- (1) $\iff$ (2)
  True by 3.2.1

- (1) $\implies$ (3)
  Let $\mathcal{O} \subseteq \mathbb{R}^m$ be open, and without loss of generality, $f^{-1}(\mathcal{O}) \neq \phi$.

  Let $a \in f^{-1}(\mathcal{O})$ so that $f(a) \in \mathcal{O}$. Hence, there is $r > 0$ such that $B_r(f(a)) \subseteq \mathcal{O}$. By continuity, there is $\delta > 0$ such that

  $$\begin{aligned} & f(B_\delta(a)) \subseteq B_r(f(a)) \subseteq \mathcal{O} \\ \implies & B_\delta(a) \subseteq f^{-1}(\mathcal{O}) \end{aligned}$$

  Hence, $\mathcal{O}$ is open.

- (3) $\implies$ (1)
  Fix $a \in S$ and let $\varepsilon > 0$. As $B_\varepsilon(f(a))$ is open in $\mathbb{R}^m$, we have $f^{-1}(B_\varepsilon(f(a)))$ is open in $\mathbb{R}^n$. But $a \in f^{-1}(B_\varepsilon(f(a)))$, and so, there is $\delta > 0$ such that

  $$\begin{aligned} & B_\delta(a) \subseteq f^{-1}(B_\varepsilon(f(a))) \\ \implies & f(B_\delta(a)) \subseteq B_\varepsilon(f(a)) \end{aligned}$$

  Hence, f is continuous at a for all $a \in S$.

$\square$

This theorem gives us a huge simplification. Recall that $x \longrightarrow y$ iff $\Pi_i(x) \longrightarrow \Pi_i(y)$ for all i. Now consider some $\{x_p\}$ with $x_p \longrightarrow a$. We have

$$f(x_p) \longrightarrow f(a) \iff \Pi_i(f(x_p)) \longrightarrow \Pi_i(f(a)) \, \forall \, i$$

That is, f is continuous iff f is continuous coordinate wise! Hence, for talking about continuity, it is enough to discuss real valued functions rather than $f : \mathbb{R}^n \to \mathbb{R}^m$ for arbitrary m.

3.3 Examples

Example 3.3.1

Consider the function,

$$f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$$
$$f(x,y) = \frac{2xy}{x^2+y^2}$$

Consider the line L_1 defined by $y = 0$ and approach $(0,0)$ from the right $(x \longrightarrow 0^+)$. We have,

$$f\big|_{L_1} \equiv 0 \implies \lim_{(x,y)\to 0 \text{ along } L_1} f = \lim_{n\to\infty} f\left(0, \frac{1}{n}\right) = 0$$

Now consider the line L_2 defined by $x = y$. We have,

$$f\big|_{L_2} \equiv 1 \implies \lim_{(x,y)\to 0 \text{ along } L_2} f = \lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = 1$$

Hence, $\lim\limits_{(x,y)\to(0,0)} f$ does not exist.

Note: The approach in the above example is often useful for showing non-existence of limits, or that a function is not continuous.

Example 3.3.2

We wish to compute $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3}{x^2+y^2}$. We have, for all $(x,y) \neq (0,0)$,

$$\left|\frac{x^3}{x^2+y^2}\right| \le \left|\frac{x^3}{x^2}\right| = |x| \le \|(x,y)\|$$

and because $\|(x,y)\|$ goes to 0 as $(x,y) \longrightarrow (0,0)$, we get

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$$

Hence, the function

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2}, \ (x,y) \neq (0,0) \\ 0, \ (x,y) = (0,0) \end{cases}$$

is continuous at $(0,0)$.

Example 3.3.3

Consider $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2}$. We have $(x^2+y^2) \longrightarrow 0$ as $(x,y) \to (0,0)$, and hence, we get

$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2} = \lim_{t\to 0} \frac{\sin t}{t} = 1$$

Exercise. Let $S \subseteq \mathbb{R}^n, a \in S'$ and $f, g : S \to \mathbb{R}$. Suppose $\lim\limits_{x \to a} f = \alpha, \lim\limits_{x \to a} g = \beta$ exist. Show that:

(i) $\lim\limits_{x \to a}(rf + g) = r\alpha + \beta, \forall\, r \in \mathbb{R}$

(ii) $\lim_{x \to a} fg = \alpha\beta$

(iii) If $\beta \neq 0$, $\lim\limits_{x \to a} \dfrac{f}{g} = \dfrac{\alpha}{\beta}$

(iv) If $f \leq h \leq g$ for some $h : S \to \mathbb{R}$ and $\alpha = \beta$, then $\lim\limits_{x \to a} h = \alpha$

Note: Similar results hold for continuity as well, using which we get the next examples.

Example 3.3.4 (Some classes of continuous functions)

(1) The projection maps $\Pi_i : \mathbb{R}^n \to \mathbb{R}$ are continuous.

(2) $x_i \in \mathbb{R}[x_1, \ldots, x_n]$ is continuous for all i.

(3) $x_i^2 \in \mathbb{R}[x_1, \ldots, x_n]$ is continuous for all i.

(4) All monomials in $\mathbb{R}[x_1, \ldots, x_n]$ are continuous.

(5) Any $p \in \mathbb{R}[x_1, \ldots, x_n]$ is continuous.

(6) $\frac{p}{q}$ is continuous at $a \in \mathbb{R}^n$, where $p, q \in \mathbb{R}[x_1, \ldots, x_n]$ and $q(a) \neq 0$.