Lecture 7

7.1 Schwarz Theorem

In the previous lecture, we discussed the notion of partial derivatives. In general, the partial derivatives depend on the order of differentiation. However, using Clairaut's Theorem, we found a necessary when $f_{xy} = f_{yx}$. Now, we conclude that discussion by the following result.

Theorem 7.1.1 (Schwarz)

Let $(a,b) \in \mathcal{O}_2$ and $f : \mathcal{O}_2 \to \mathbb{R}$. Suppose f_x, f_y, and f_{xy} exist on $\mathcal{O}_2$. If f_{xy} is continuous at (a,b), then f_{yx} exists in a neighbourbood of (a,b) and $f_{xy}(a,b) = f_{yx}(a,b)$.

Proof. Just as before, we take $(a,b) = (0,0)$. From the proof of Clairaut's Theorem, we have $F(h,k) = f_{xy}(c_1, c_2)$ for some $0 < c_1 < h$ and $0 < c_2 < k$. By continuity of f_{xy} at $(0,0)$, for any $\epsilon > 0$ there exists $h_\epsilon, k_\epsilon > 0$, such that

$$\mid f_{xy}(u,v) - f_{xy}(0,0) \mid < \epsilon \;\; \forall \, (u,v) \in [0, h_\epsilon] \times [0, k_\epsilon]$$

But then $\mid F(h,k) - f_{xy}(0,0) \mid < \epsilon \;\; \forall \, (u,v) \in [0, h_\epsilon] \times [0, k_\epsilon]$, that is, F is continuous at $(0,0)$ with limit $f_{xy}(0,0)$ at $(0,0)$.
As $f_y(h,0)$ exists for h sufficiently small, $\lim_{k \to 0} F(h,k)$ exists for h sufficiently small. Thus, by continuity of F at $(0,0)$,

$$\lim_{h \to 0} \lim_{k \to 0} F(h,k) \text{ exists and is equal to } \lim_{(h,k) \to (0,0)} F(h,k)$$

Thus, $f_{yx}(0,0)$ exists and is equal to $f_{xy}(0,0)$ □

This is a slightly more useful version of Clairaut's Theorem. However, in many applications (say, partial differential equations), we work with C^2 or even C^∞ functions, in which case both of these hold automatically.

Exercise: Formulate and prove a similar result for higher order derivatives. In particular, provide a sufficient condition for $f : \mathcal{O}_n \to \mathbb{R}$ so that

$$\frac{\partial^n f}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_m}} = \frac{\partial^n f}{\partial x_{\sigma(i_1)} \partial x_{\sigma(i_2)} \ldots \partial x_{\sigma(i_m)}}$$

over $\mathcal{O}_n$ for any permutation σ of the elements $\{i_1, i_2, \ldots, i_n\}$.

7.2 Partial and Total Derivatives

We will now see that the partial derivatives provide an effective way of proving the existence and computing the total derivatives of a function $f : \mathcal{O}_n \to \mathbb{R}^m$. In this lecture and the next, we will develop the relations between partial and total derivatives by a series of results.

Definition 7.2.1 ▶ Jacobian Matrix

For a function $f = (f_1, f_2, \dots, f_m) : \mathcal{O}_n \to \mathbb{R}^m$, if all the partial derivatives $\frac{\partial f_i}{\partial x_j}$ at $a \in \mathcal{O}_n$, we define the Jacobian of the function at a by the $m \times n$ matrix,

$$J_f(a) = \left(\frac{\partial f_i}{\partial x_j}(a)\right)_{m\times n}$$

Theorem 7.2.1

Consider a function $f = (f_1, f_2, \dots, f_m) : \mathcal{O}_n \to \mathbb{R}^m$ differentiable at $a \in \mathcal{O}_n$. Then all the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist at a. In particular, for f differentiable at a, we have,

$$(Df)(a) = J_f(a) = \left(\frac{\partial f_i}{\partial x_j}(a)\right)_{m\times n}$$

Proof. Without loss of generality, we take $m = 1$, and let $a = (a_1, a_2, \dots, a_n)$. Fix an arbitrary index $i \in \{1, 2, \dots, n\}$. We define $\eta_i : [a_i - \epsilon, a_i + \epsilon] \to \mathbb{R}^n$, defined by

$$\eta_i(t) = (a_1, \dots, a_{i-1}, t, a_{t+1}, \dots, a_n) = a + (t - a_i)e_i$$

As $\mathcal{O}_n$ is open and η_i is continuous, we can find ϵ small such that $f([[a_i - \epsilon, a_i + \epsilon]]) \subseteq \mathcal{O}_n \subseteq \mathbb{R}^n$. Evidently, η_i is differentiable and $(D\eta_i) = [0, \dots, 1, \dots, 0]^t = e_i^t$ over $[a_i - \epsilon, a_i + \epsilon]$. Now, by the definition of partial derivatives, $D(f \circ \eta_i)(a_i) = f_{x_i}(a)$.
Again, by chain rule, as f is differentiable at a, $D(f \circ \eta_i)(a_i) = f_{x_i}(a)$ exists, and

$$\begin{aligned} D(f \circ \eta_i)(a_i) &= Df(\eta_i(a_i)) \cdot D\eta_i(a_i) \\ \implies f_{x_i}(a) &= Df(a) \cdot e_i^t = [Df(a)]_i \end{aligned}$$

As the index i was arbitrary to begin with, this completes the proof. □

This theorem proves that differentiability of a function implies the existence of its partial derivatives, and gives the form of the derivative in the standard basis. But it is often quite elaborate and laborious to prove that a function is differentiable, whereas computation of the partial derivatives is much more straightforward. In the next lecture, we formulate a sufficient condition for differentiability of a function based on its partial derivatives.