Lecture 8

8.1 A kind of converse of Theorem 7.2.1

As we have seen in previous lecture, the differentiability of a function gives an explicit expression for derivative with the existence of partials. In this lecture, we will prove a sufficient condition for differentiability based on its partials, which will be our final reduction for derivatives.

Theorem 8.1.1 (Final Reduction)

Let $f : \mathcal{O}_n \to \mathbb{R}^m$ and $a \in \mathcal{O}_n$. Suppose, all partial derivatives $\frac{\partial f_i}{\partial x_j}$ exists on $\mathcal{O}_n$ and continuous at $a \in \mathcal{O}_n$. Then,

$$Df(a) = J_f(a) = \left(\frac{\partial f_i}{\partial x_j}(a) \right)_{m \times n}$$

Proof. Without loss of generality, we take $a = (0, \dots, 0) \in \mathcal{O}_n$ and $m = 1$.

Let's do some back calculation: We already "know",

$$L = J_f(a) = \begin{pmatrix} f_{x_1}(0) & \cdots & f_{x_n}(0) \end{pmatrix} \text{ and } \boxed{Lh = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(0) \ \forall \ h \in \mathbb{R}^n}$$

which we use in the following claim,

Claim

$$\frac{1}{\|h\|} |f(h) - f(0) - Lh| \to 0 \text{ as } h \to 0$$

Proof. Simply calculating,

$$\frac{1}{\|h\|} |f(h) - f(0) - Lh| = \frac{1}{\|h\|} \left| f(h) - f(0) - \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(0) \right|$$

For every i, we define, $\hat{h}_i = (h_1, \dots, h_i, \underbrace{0, \dots, 0}_{n-r})$ and $\hat{h}_0 = 0$. Then,

$$\begin{aligned} f(h) - f(0) &= \Big(f(\hat{h}_1) - f(\hat{h}_0) \Big) + \Big(f(\hat{h}_2) - f(\hat{h}_1) \Big) + \cdots + \Big(f(\hat{h}_n) - f(\hat{h}_{n-1}) \Big) \\ &= \sum_{i=1}^{n} \Big(f(\hat{h}_i) - f(\hat{h}_{i-1}) \Big) \end{aligned}$$

which implies,

$$\begin{aligned}\left|f(h)-f(0)-\sum_{i=1}^{n}h_i\frac{\partial f}{\partial x_i}(0)\right| &= \left|\sum_{i=1}^{n}\left(f(\hat{h}_i)-f(\hat{h}_{i-1})-h_i\frac{\partial f}{\partial x_i}(0)\right)\right| \\ &= \left|\sum_{i=1}^{n}\underbrace{h_i\frac{\partial f}{\partial x_i}\left(\hat{h}_{i-1}+c_ie_i\right)}_{\text{By MVT, as explained below}}-h_i\frac{\partial f}{\partial x_i}(0)\right|\end{aligned}$$

For a fixed h, we fix $i\in[n]$. And we consider the map, $\eta_i:(h_i-\epsilon,h_i+\epsilon)\to\mathbb{R}$

$$\begin{tikzcd} t \arrow[r, mapsto] \arrow[rr, bend left, "\eta_i"] & \hat{h}_{i-1}+te_i \arrow[r, mapsto] & f(\hat{h}_{i-1}+te_i) \end{tikzcd}$$

defined by, $\eta_i(t)=f(\hat{h}_{i-1}+te_i)$. Clearly, η_i is differentiable on $(0,h_i)$ and continuous on $[0,h_i]$. Then, by Mean Value Theorem,

$$\underbrace{\eta_i(h_i)}_{f(\hat{h}_i)}-\underbrace{\eta_i(0)}_{f(\hat{h}_{i-1})}=\eta_i'(c_i)h_i=f_{x_i}(\hat{h}_{i-1}+c_ie_i)h_i \qquad \text{(for some } c_i\in(0,h_i))$$

Now, observe that, as $h\to 0, \hat{h}_{i-1}+c_ie_i\to 0$ which in turn implies, $f_{x_i}(\hat{h}_{i-1}+c_ie_i)\to f_{x_i}(0)$. Therefore,

$$\begin{aligned}\frac{1}{\|h\|}\left|f(h)-f(0)-Lh\right| &= \frac{1}{\|h\|}\left|\sum_{i=1}^{n}h_i\frac{\partial f}{\partial x_i}\left(\hat{h}_{i-1}+c_ie_i\right)-h_i\frac{\partial f}{\partial x_i}(0)\right| \\ &\leq \frac{1}{\|h\|}\sum_{i=1}^{n}|h_i|\left|\frac{\partial f}{\partial x_i}\left(\hat{h}_{i-1}+c_ie_i\right)-\frac{\partial f}{\partial x_i}(0)\right| && \text{(Triangle inequality)} \\ &\leq \sum_{i=1}^{n}\left|\frac{\partial f}{\partial x_i}\left(\hat{h}_{i-1}+c_ie_i\right)-\frac{\partial f}{\partial x_i}(0)\right| && (\text{as } \|h\|\geq|h_i|\ \forall\ i) \\ &\longrightarrow 0 \text{ as } h\to 0\end{aligned}$$

□

And, this completes the proof. □

With Theorem 8.1.1, computation of derivative is much easier when we are in favorable situation. Note that,

(i) If f is differentiable at a then $\frac{\partial f_i}{\partial x_j}(a)$ exists for all i,j and $Df(a)=J_f(a)$.

(ii) If $\frac{\partial f_i}{\partial x_j}$ is continuous at a then f is differentiable and $Df(a)=J_f(a)$.

The gap between (i) and (ii) is the continuity of partials, which is removable.

8.2 Examples

We conclude the lecture with some instructive examples.

Example 8.2.1 (Differentiable but discontinuous)

Take,

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), & (x,y) \neq 0 \\ 0, & (x,y) = 0 \end{cases}$$

Then,

$$\begin{aligned} |f(x,y) - f(0,0)| &= \left|x^2+y^2\right| \left|\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)\right| \\ &\leq (x^2+y^2) = \|(x,y)\|^2 \end{aligned}$$

implies that f is continuous at $(0,0)$. For all $(x,y) \neq (0,0)$, the partials of f,

$$\begin{aligned} f_x(x,y) &= 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) \\ f_y(x,y) &= 2y\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{y}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) \end{aligned}$$

And at $(0,0)$,

$$\begin{aligned} f_x(0,0) &= \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = 0 \\ f_y(0,0) &= \lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t} = 0 \end{aligned}$$

Also,

$$\frac{1}{\sqrt{h^2+k^2}}\left|f(h,k) - f(0,0) - \begin{pmatrix}0 & 0\end{pmatrix}\begin{pmatrix}h \\ k\end{pmatrix}\right| = \sqrt{h^2+k^2}\left|\sin\left(\frac{1}{\sqrt{h^2+h^2}}\right)\right| \leq \|(h,k)\|$$

which shows that, f is differentiable at $(0,0)$ and $Df(0,0) = \begin{pmatrix}0 & 0\end{pmatrix}$. But, f_x and f_y are not continuous at $(0,0)$!

So, even if a function is differentiable at some point, its partials may still not be continuous there!

Example 8.2.2 (Exercise)

Take,

$$f(x,y) = \begin{cases} x^{\frac{4}{3}}\sin\left(\frac{y}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

- Show that,
  1. f is differentiable on $\mathbb{R}^2$.
  2. f_x and f_y exist and continuous on $\mathcal{O}_2 = \left\{(x,y) \in \mathbb{R}^2 : x \neq 0\right\}$.
  3. f_x is not continuous at $(0,y)$ for all $y \neq 0$.
- Discuss the nature of continuity of f at the origin.

Example 8.2.3

Let, $f : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by,

$$f(x, y) = (x + 2y + 3z, xyz, \cos x, \sin x)$$

Then, the at (x, y, z)

$$J_f(x, y, z) = \begin{pmatrix} 1 & 2 & 3 \\ yz & zx & xy \\ -\sin x & 0 & 0 \\ \cos x & 0 & 0 \end{pmatrix}$$

which has every entry continuous, thus

$$J_f(x, y, z) = Df(x, y, z)$$