Lecture 9

9.1 Directional Derivatives

We now introduce another extension of one-dimensional derivative, called the Directional Derivative.

Definition 9.1.1 ▶ Directional Derivative

Let $u \in \mathbb{R}^n$ (indicates a direction) be a unit vector and $f : \mathcal{O}_n \to \mathbb{R}$ be a scalar-valued function. Take, $a \in \mathcal{O}_n$, the directional derivative of f at a in the direction of u is defined as

$$(D_u f)(a) := \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t}$$

the rate of change of f at a in the direction of u, provided the limit exists.

Note that $(D_{e_i} f)(a) = \frac{\partial f}{\partial x_i}(a)$, i.e., the partial derivatives are directional derivatives along the standard basis vectors.

Now, consider the following

$$t \overset{\phi}{\longmapsto} a + tu \overset{f}{\longmapsto} f(a+tu), \qquad t \overset{\eta(t) = f(a+tu)}{\longmapsto} f(a+tu)$$

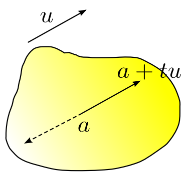

Figure 9.1: $t \mapsto a + tu$

Take, $\eta : (-\varepsilon, \varepsilon) \to \mathbb{R}$ as $\eta(t) := f(a + tu)$. If f is differentiable at a then η is differentiable at 0 (by chain rule) iff f has directional derivative at a along u and,

$$\begin{aligned}
\eta'(0) &= (D_u f)(a) \\
&= Df(\phi(0)) \cdot \underbrace{(D\phi)(0)}_{u} \\
&= \begin{pmatrix} \frac{\partial f}{\partial x_i}(a) & \cdots & \frac{\partial f}{\partial x_n}(a) \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}
\end{aligned}$$

So,

$$\boxed{\eta'(0) = \sum_{i=1}^{n} u_i \frac{\partial f}{\partial x_i}(a) = (D_u f)(a)}$$

9.2 Gradient

Definition 9.2.1 ▶ Gradient

Given a scalar-valued function (a.k.a., scalar field) $f : \mathcal{O}_n \to \mathbb{R}$ and $x \in \mathcal{O}_n$, the Gradient of f at x is defined as

$$\nabla f(x) := \langle f_{x_1}(x), \dots, f_{x_n}(x) \rangle$$

the dual of the total derivative, i.e., $Df(x)^t$, provided that all the partials f_{x_i} exists at x.

Observe that, for a differentiable function $f : \mathcal{O}_n \to \mathbb{R}$ at a

$$\begin{aligned} (D_u f)(a) &= (\nabla f)(a) \cdot u \\ &= \|(\nabla f)(a)\| \cos\theta_u \qquad \text{(where, } \theta_u \text{ is angle between } (\nabla f)(a) \text{ and } u\text{)} \end{aligned}$$

which tells us, The steepest slope is achieved when $\theta_u \in \{0, \pi\}$, i.e., when u points along or opposite to the direction of $(\nabla f)(a)$ that means $\max (D_u f)(a)$ is attained at $\underbrace{u = \dfrac{(\nabla f)(a)}{\|(\nabla f)(a)\|}}_{\text{provided } \|(\nabla f)(a)\| \neq 0}$ (direction of the steepest slope). Hence, we get the following theorem,

Theorem 9.2.1

Let $f : \mathcal{O}_n \to \mathbb{R}$ be a differentiable function at $a \in \mathcal{O}_n$ and suppose $(\nabla f)(a) \neq 0$ then the vector $(\nabla f)(a)$ points in the direction of the greatest increment of f at a with the greatest rate $\|(\nabla f)(a)\|$

9.3 Examples

Example 9.3.1

Find the directional derivative of $f(x, y, z) = x^2yz$ along $\langle 1, 1, -1 \rangle$ at $a = (1, 1, 0)$.

Solution. We have the unit vector $u = \frac{\langle 1,1,-1 \rangle}{\|\langle 1,1,-1 \rangle\|} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$. So,

$$\begin{aligned} (D_u f)(a) &= (\nabla f)(a) \cdot u \\ &= \begin{pmatrix} 2xyz & x^2z & x^2y \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \end{aligned}$$

Therefore, $(D_u f)(1, 1, 0) = -\frac{1}{\sqrt{3}}$ and hence maximum value of $(D_u f)(1, 1, 0)$ is $\|(\nabla f)(1, 1, 0)\|$ along the unit vector $\langle 0, 0, 1 \rangle$.

Example 9.3.2

Take,

$$f(x, y) = \begin{cases} \frac{x^2y}{x^2+y^2} & (x, y) = (0, 0) \\ 0 & (x, y) \neq (0, 0) \end{cases}$$

then $|f(x, y) - f(0, 0)| = \left| \frac{x^2y}{x^2+y^2} \right| \leq |y| \leq \|(x, y)\|$ implies that f is continuous at $(0, 0)$. Now, fix

$u = \langle u_1, u_2 \rangle$ with $\|u\| = 1$. We get,

$$\begin{aligned} (D_u f)(0,0) &= \lim_{t \to 0} \frac{f(tu) - 0}{t} \\ &= \lim_{t \to 0} \frac{1}{t} \cdot \frac{t u_1^2 u_2}{1} \\ &= u_1^2 u_2 \neq 0 && \text{(Because, } u \text{ is an unit vector)} \end{aligned}$$

If we assume f to be differentiable then $(\boldsymbol{\nabla} f)(0,0) \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \neq u_1^2 u_2$ which is a contradiction!

Example 9.3.2 shows that, the existence of all partial and directional derivatives at a point fails to imply differentiability at that point.