Lecture 10

Example 10.0.1 (Exercise)

Take the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as,

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(i) Prove that, f is continuous at $(0,0)$.

(ii) Find $(D_u f)(0,0)\ \forall\ u$.

(iii) Prove that, f is not differentiable.

10.1 Extension of MVT to Several Variables

In Analysis I, you learned about the Mean Value Theorem (MVT) for functions of a single variable. Now, we extend this concept to several variables in the context of multivariable calculus.

Theorem 10.1.1 (Multivariate MVT)

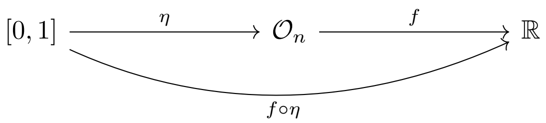
Let $\mathcal{O}_n \subseteq \mathbb{R}^n$ be an open and convex set, and let $f : \mathcal{O}_n \to \mathbb{R}$ be a differentiable function. For any two points $a, b \in \mathcal{O}_n$ define the line segment

$$L_{a,b} := \{tb + (1-t)a : t \in [0,1]\}$$

Then, there exists a point $c \in L_{a,b}$ such that,

$$f(b) - f(a) = (\boldsymbol{\nabla} f)(c) \cdot (b-a) = \langle f_{x_1}(c), \ldots, f_{x_n}(c)\rangle \cdot \langle (b_1 - a_1), \ldots, (b_n - a_n)\rangle$$

Proof. We consider the function $\eta : [0,1] \to \mathcal{O}_n$



defined by $\eta(t) = (1-t)a + tb$. This function is differentiable, and its derivative is

$$\eta'(t) = \begin{pmatrix} b_1 - a_1 \\ \vdots \\ b_n - a_n \end{pmatrix}$$

By applying the standard Mean Value Theorem to the composition $f \circ \eta$, there exists $t_0 \in (0,1)$ such that,

$$f(\eta(1)) - f(\eta(0)) = (f \circ \eta)'(t_0) = Df(\eta(t_0)) \cdot D\eta(t_0)$$

Expanding the dot product, we have,

$$f(b) - f(a) = \begin{pmatrix} f_{x_1}(\eta(t_0)) & f_{x_2}(\eta(t_0)) & \cdots & f_{x_n}(\eta(t_0)) \end{pmatrix} \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_n - a_n \end{pmatrix}$$

Simplifying further, we obtain,

$$f(b) - f(a) = \langle f_{x_1}(\eta(t_0)), \cdots, f_{x_n}(\eta(t_0)) \rangle \cdot \langle (b_1 - a_1), \cdots, (b_n - a_n) \rangle$$

This expression can be rewritten as,

$$f(b) - f(a) = (\boldsymbol{\nabla} f)(\eta(t_0)) \cdot (b - a)$$

Hence, there exists $c = \eta(t_0) \in L_{a,b}$ such that $f(b) - f(a) = (\nabla f)(c) \cdot (b - a)$. □

10.2 More Partials and Chain Rules

In this section, we further explore the chain rule for differentiable functions of several variables. Consider two functions f and g as following,

$$\mathcal{O}_n \xrightarrow{\quad f \quad} \mathcal{O}_m \xrightarrow{\quad g \quad} \mathbb{R}^p$$

Assuming that f is differentiable at $a \in \mathcal{O}_n$ and g is differentiable at $b = f(a) \in \mathcal{O}_m$ the chain rule states that the derivative of the composite function $g \circ f$ is given by,

$$\begin{aligned} \underbrace{D(g \circ f)(a)}_{\mathbb{R}^n \to \mathbb{R}^p} &= (Dg)(f(a)) \cdot (Df)(a) \\ &= \underbrace{Dg(b)}_{\mathbb{R}^m \to \mathbb{R}^p} \cdot \underbrace{Df(a)}_{\mathbb{R}^n \to \mathbb{R}^m} \end{aligned}$$

This can be expressed in matrix form as,

$$J_{g \circ f}(a)_{p \times n} = J_g(f(a))_{p \times m} \cdot J_f(a)_{m \times n} \tag{10.1}$$

Moreover, if we consider the function components in each individual coordinates as

- $g \circ f = ((g \circ f)_1, (g \circ f)_2, \ldots, (g \circ f)_p)$
- $g = (g_1, g_2, \ldots, g_p)$
- $f = (f_1, f_2, \ldots, f_m)$.

Then, the $(i,j)^{\text{th}}$ entry of both sides of (10.1) would become,

$$\frac{\partial (g \circ f)_i}{\partial x_j}(a) = \begin{pmatrix} \frac{\partial g_i}{\partial y_1}(f(a)) & \frac{\partial g_i}{\partial y_2}(f(a)) & \cdots & \frac{\partial g_i}{\partial y_m}(f(a)) \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_j}(a) \\ \frac{\partial f_2}{\partial x_j}(a) \\ \vdots \\ \frac{\partial f_m}{\partial x_j}(a) \end{pmatrix}$$

which is our familiar chain rule for partial derivatives,

$$\boxed{\frac{\partial (g \circ f)_i}{\partial x_j}(a) = \sum_{k=1}^{m} \frac{\partial g_i}{\partial y_k}(b) \cdot \frac{\partial f_k}{\partial x_j}(a)}$$

where, $b = f(a)$.

Furthermore, if we define $y_k = f_k(x_1, x_2, \ldots, x_n)$ and $z_i = g_i(y_1, y_2, \ldots, y_m)$ we can express the chain rule for partial derivatives as,

$$\boxed{\frac{\partial z_i}{\partial x_j}(x) = \sum_{k=1}^{m} \frac{\partial z_i}{\partial y_k}(y) \cdot \frac{\partial y_k}{\partial x_j}(x)}$$

Remark. In addition, the chain rule can be applied in the context of a composition map with respect to a parameter t. For functions $f : \mathcal{O}_n \to \mathbb{R}$, $\eta : \mathcal{O}_1 \to \mathcal{O}_n$ and $z = f \circ \eta$ as shown below,

$$\mathcal{O}_1 \xrightarrow{\quad \eta \quad} \mathcal{O}_n \xrightarrow{\quad f \quad} \mathbb{R}$$

$$t \xmapsto{\quad \eta \quad} (x_1(t), \ldots, x_n(t)) \xmapsto{\quad f \quad} f(x_1(t), \ldots, x_n(t))$$

$$t \xmapsto{\quad z = f \circ \eta \quad} f(x_1(t), \ldots, x_n(t))$$

the chain rule states,

$$\boxed{\frac{\mathrm{d}z}{\mathrm{d}t} = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \cdot \frac{\mathrm{d}x_k}{\mathrm{d}t}}$$

If we treat f as a function of t, the same can be written as,

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} \cdot \frac{\mathrm{d}x_k}{\mathrm{d}t}}$$