Lecture 16

16.1 Riemann-Darboux Integration

Definition 16.1.1 ▶ Volume of a closed/open ball

We define the volume of a closed ball $B^n = \prod_{i=1}^{n} [a_i, b_i]$ as $\text{Vol}(B^n) = \prod_{i=1}^{n} (b_i - a_i)$. We also define the volume of the open set $O^n = \prod_{i=1}^{n} (a_i, b_i)$ to be equal to that of B^n, i.e., $\text{Vol}(O^n) = \text{Vol}(B^n)$.

We now introduce some notation. Fix $i \in \{1, 2, \ldots, n\}$. We define a partition of the i^{th} interval $[a_i, b_i]$ as

$$P_i : a_i = a_{i,0} < a_{i,1} < \cdots < a_{i,n_i} = b_i$$

and the intervals of its partition as

$$I_{i,t} = [x_{i,t-1}, x_{i,t}], \forall\ 1 \le t \le n_i$$

and set

$$B^n_{(t_1,t_2,\ldots,t_n)} = B_\alpha = [x_{1,t_1-1}, x_{1,t_1}] \times \cdots \times [x_{n,t_n-1}, x_{n,t_n}] = I_{1,t_1} \times \cdots \times I_{n,t_n}$$

where α is chosen from the indexing set $\Lambda(P) = \{\alpha = (t_1, \ldots, t_n) \mid 1 \le t_i \le n_i, i = 1, \ldots, n\}$.

Note

1. $B^n = \bigcup_{\alpha \in \Lambda(P)} B^n_\alpha$
2. $\text{Vol}(B^n) = \sum_{\alpha \in \Lambda(P)} \text{Vol}(B^n_\alpha)$

We call $\mathscr{P}(B) = \{P_1 \times \cdots \times P_n \mid P_i \in \mathscr{P}[a_i, b_i]\}$ the set of all partitions of B^n.

Definition 16.1.2 ▶ Refinement of Partitions

Given $P = \prod_{i=1}^{n} P_i$ and $\widetilde{P} = \prod_{i=1}^{n} \widetilde{P}_i$ with $P, \widetilde{P} \in \mathscr{P}[a, b]$, then $\widetilde{P}$ is called a refinement of P if $\widetilde{P}_i \supset P_i\ \forall i = 1, 2, \ldots, n$.

Theorem 16.1.1

Let f be a bounded function over B^n. Let $P, \widetilde{P} \in \mathscr{P}(B^n)$ and $\widetilde{P} \supset P$. Then

$$L(f, P) \le L(f, \widetilde{P}) \le U(f, \widetilde{P}) \le U(f, P)$$

Proof. Note that $L(f, \widetilde{P}) \leq U(f, \widetilde{P})$ follows directly from the fact that $m_\alpha(\widetilde{P}) \leq M_\alpha(\widetilde{P}) \; \forall \widetilde{P} \in \mathscr{P}[a, b]$, where $m_\alpha = \inf_{B_\alpha^n} f$ and $M_\alpha = \sup_{B_\alpha^n} f$. ☐

Corollary (Inequality of upper and lower sums)

For all $P, \widetilde{P} \in \mathscr{P}(B^n)$, the following inequality holds.

$$m \times \text{Vol}(B^n) \leq L(f, P) \leq U(f, \widetilde{P}) \leq M \times \text{Vol}(B^n)$$

We denote $\mathscr{B}(A) = \{f : A \to \mathbb{R} \mid \sup_A |f| < \infty\}$ as the set of all bounded functions over A for any $A \subseteq \mathbb{R}^n$.

Definition 16.1.3 ▶ Upper and Lower Darboux Integrals

For $f \in \mathscr{B}(B^n)$, we define

$$\overline{\int}_{B^n} f = \inf_{P \in \mathscr{P}(B^n)} U(f, P) \text{ and } \underline{\int}_{B^n} f = \sup_{P \in \mathscr{P}(B^n)} L(f, P)$$

as the Upper and Lower Darboux Integrals, respectively.

We have $L(f, P) \leq U(f, P')$ for all $P, P' \in \mathscr{P}(B^n)$ by taking the common refinement $\widehat{P} = P \cup P'$. Hence,

$$\underline{\int}_{B^n} f \leq \overline{\int}_{B^n} f$$

Definition 16.1.4 ▶ Darboux Integral

Let $f \in \mathscr{B}(B^n)$. f is said to be Riemann-Darboux Integrable or Riemann Integrable or just Integrable if

$$\underline{\int}_{B^n} f = \overline{\int}_{B^n} f$$

In this case, we introduce the notation,

$$\int_{B^n} f \, \mathrm{d}V = \int_{B^n} f(x_1, \ldots, x_n) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n = \underline{\int}_{B^n} f = \overline{\int}_{B^n} f$$

At this point, the notation $\int_{B^n} f(x_1, \ldots, x_n) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n$ **does not** indicate repeated integration, but we will see that it represents repeated integration for "nice" functions.