# Lecture 17

## 17.1 Properties of Riemann-Darboux Integration

In the previous lecture, we introduced the Riemann-Darboux Integral. In this lecture, we will explore some important properties of this integral, starting with a characterization.

**Theorem 17.1.1 (Classification of Riemann Integrable Functions)**

Let $f \in \mathscr{B}(B^n)$. Then $f \in \mathscr{R}(B^n)$ if and only if for every $\epsilon > 0$, there exists a partition $P \in \mathscr{P}(B^n)$ of $B^n$ such that

$$(0 \leq)\ U(f, P) - L(f, P) < \epsilon$$

*Proof.* $\implies$ Suppose $f \in \mathscr{R}(B^n)$. Then we have

$$\overline{\int} f - \underline{\int} f = 0$$

Thus,

$$\begin{aligned} 0 = \overline{\int} f - \underline{\int} f &= \inf_{P \in \mathscr{P}(B^n)} U(f, P) - \sup_{P \in \mathscr{P}(B^n)} L(f, P) \\ &= \inf_{P \in \mathscr{P}(B^n)} (U(f, P) - L(f, P)) \end{aligned}$$

Hence for all $\epsilon > 0$, there exists a partition $P \in \mathscr{P}(B^n)$ such that $U(f, P) - L(f, P) < \epsilon$.

$\impliedby$ Conversely, assume that for every $\epsilon > 0$, there exists a partition $P \in \mathscr{P}(B^n)$ such that $U(f, P) - L(f, P) < \epsilon$. We want to show that $\overline{\int} f = \underline{\int} f$. Since $U(f, P) \geq \overline{\int} f$ and $L(f, P) \leq \underline{\int} f$, it follows that for all $\epsilon > 0$,

$$0 \leq \overline{\int} f - \underline{\int} f < \epsilon$$

This implies $\overline{\int} f = \underline{\int} f$, showing that $f \in \mathscr{R}(B^n)$.

□

**Exercise.** Let $f, g \in \mathscr{R}(B^n)$. Then show that,

- $|f| \in \mathscr{R}(B^n)$ and $\left| \int_{B^n} f \right| \leq \int_{B^n} |f|$

- $\mathscr{R}(B^n)$ is a $\mathbb{R}-$algebra by showing the following,
  - (i) For any $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g \in \mathscr{R}(B^n)$
  - (ii) $fg \in \mathscr{R}(B^n)$

Next, we demonstrate the Riemann integrability of a class of "nice" functions (continuous). However, before proceeding, let's introduce the concepts of the diameter of a set and the mesh of a partition.

**Definition 17.1.1 ▶ Diameter of a set**

For a set $A \subseteq \mathbb{R}^n$, the diameter $d(A)$ is defined as $d(A) = \sup\{|x - y| \mid x, y \in A\}$.

**Exercise.** Show that $d(B^n) = \max\{\|v_i - v_j\| \mid v_i, v_j \text{ are vertices of } B^n\}$

**Definition 17.1.2 ▶ Mesh of a Partition**

For a partition $P \in \mathscr{P}(B^n)$, the mesh $\|P\|$ is defined as $\|P\| = \max\{d(B^n_\alpha) \mid \alpha \in \Lambda(P)\}$.

**Theorem 17.1.2 (Continuous Functions are Riemann Integrable)**

The set of all continuous functions over $B^n$ is contained in $\mathscr{R}(B^n)$, i.e.,

$$C(B^n) \subset \mathscr{R}(B^n)$$

*Proof.* Let $f \in C(B^n)$. Since $f$ is uniformly continuous, for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in B^n$ with $\|x - y\| < \delta$, we have,

$$|f(x) - f(y)| < \underbrace{\frac{\epsilon}{2\,\mathrm{Vol}(B^n)}}_{\text{Call it } \tilde{\epsilon}} \tag{17.1}$$

Let $P \in \mathcal{P}(B^n)$ be a partition such that $\|P\| < \delta$. For each $\alpha \in \Lambda(P)$, let $a_\alpha \in B^n_\alpha$. Then $\|x - a_\alpha\| < \delta$ for all $x \in B^n_\alpha$. It follows from the uniform continuity condition (17.1) that,

$$|f(x) - f(a_\alpha)| < \tilde{\epsilon}$$
$$\text{i.e., } f(a_\alpha) - \tilde{\epsilon} < f(x) < f(a_\alpha) + \tilde{\epsilon} \tag{17.2}$$

Since, (17.2) holds for all $\alpha \in \Lambda(P)$, $a_\alpha \in B^n_\alpha$ and for all $x \in B^n_\alpha$ we have,

$$f(a_\alpha) - \tilde{\epsilon} \leq m_\alpha \leq M_\alpha \leq f(a_\alpha) + \tilde{\epsilon}$$

Multiplying the volumes of $B^n_\alpha$ and summing over $\Lambda(P)$, we obtain,

$$\sum_{\alpha \in \Lambda(P)} f(a_\alpha)\,\mathrm{Vol}(B^n_\alpha) - \frac{\epsilon}{2} \leq L(f, P) \leq U(f, P) \leq \sum_{\alpha \in \Lambda(P)} f(a_\alpha)\,\mathrm{Vol}(B^n_\alpha) + \frac{\epsilon}{2}$$

Thus, $U(f, P) - L(f, P) < \epsilon$, and since $\epsilon$ is arbitrary, we conclude that $f \in \mathscr{R}(B^n)$. □

Now, let's consider an important question: Does an analogue of the Fundamental Theorem of Calculus exist in higher dimensions?

In one dimension ($n = 1$), we have the useful relationship $\displaystyle\int_{[a,b]} \mathrm{d}f = f\Big|_{\partial[a,b]}$, which aids in computation.

However, this relationship becomes less practical in higher dimensions. For instance, in $n = 1$, the continuous counterpart to a sum $\sum a_n$ is the one-dimensional integral $\displaystyle\int_{B^1} f$. Similarly, in $n = 2$, the continuous analogue to a double sum $\sum a_{mn}$ is the two-dimensional integral $\displaystyle\int_{B^2} f$.

## 17.2 Iterated Integrals

Before delving deeper into the concept of integrability, let's take a brief detour to discuss the idea of a double sum.

**Definition 17.2.1 ▶ Convergence of Double Sequence**

A double sequence $\{a_{mn}\}$ converges to $a$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|a_{mn} - a| < \epsilon$ for all $m, n \geq N$

Consider the following examples,

**Example** 17.2.1

- Let's take the sequence $\{a_{mn}\}$ defined by $a_{mn} = \frac{1}{m+n}$ for all $m, n \in \mathbb{N}$. This sequence is bounded, and for $N > \frac{1}{2\epsilon}$, we have $|a_{mn} - 0| = a_{mn} < \epsilon$ for all $m, n \geq N$.
- Now consider the sequence $\{a_{mn}\}$ defined as follows,

$$a_{mn} = \begin{cases} n & \text{if } m = 1 \\ \frac{1}{m+n} & \text{otherwise} \end{cases}$$

  This sequence is also convergent but not bounded.

Recall the relation between total limit and iterated limits in double sequence,

**Theorem**

For a double sequence $\{a_{mn}\}$ if $\lim_{m,n\to\infty} a_{mn}$ exists and $\lim_{m\to\infty} a_{mn}$ exists for all $n$, then

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = \lim_{m,n\to\infty} a_{mn}$$

An important analogue of the above result is Fubini's Theorem. Computation of the Darboux integral is typically a challenging task. However, Fubini's Theorem offers a valuable approach that simplifies the computation by utilizing iterated integrals.

### Visualization

We look at slice functions along each axis, which enables us to simplify computations and apply Fubini's Theorem for efficient evaluation of multivariable integrals.

Consider a function $f : B^2 \to \mathbb{R}$. For each $x \in [a_1, b_1]$ we define a slice function $f_x : [a_2, b_2] \to \mathbb{R}$ given by $f_x(y) = f(x, y)$ for all $y \in [a_2, b_2]$. This function is obtained by fixing $x$ and slicing along the $y$-axis at that $x$-coordinate. Then an iterated integral becomes

$$\int_{[a_1,b_1]}\left(\int_{[a_2,b_2]} f_x(y)\,\mathrm{d}y\right)\mathrm{d}x$$

The question arise whether this quantity is invariant under the interchange of $x$ and $y$, i.e., we may slice $f$ along $x-$axis at $y$ to obtain $f_y : [a_1, b_1] \to \mathbb{R}$ given by $f_y(x) = f(x, y)$ for every $x \in [a_1, b_1]$ and want to investigate the equality of

$$\int_{[a_1,b_1]}\left(\int_{[a_2,b_2]} f_x(y)\,\mathrm{d}y\right)\mathrm{d}x \overset{?}{=} \int_{[a_2,b_2]}\left(\int_{[a_1,b_1]} f_y(x)\,\mathrm{d}x\right)\mathrm{d}y \overset{?}{=} \int_{B^2} f \tag{17.3}$$

In this context, we observe that a partition $P \in \mathscr{P}(B^2)$ can be decomposed into the partitions of the individual coordinates. Specifically, we have $P = P_1 \times P_2$ for the two coordinate intervals $[a_1, b_1]$ and $[a_2, b_2]$, and the corresponding indexing sets satisfy $\Lambda(P) = \Lambda(P_1) \times \Lambda(P_2)$.

Now consider the following example,

**Example** 17.2.2 (A discrepancy: Function integrable, slices not)

Let $I = [0,1]$ and $B^2 = I \times I$. Consider the function $f : B^2 \to \mathbb{R}$ given by,

$$f(x,y) = \begin{cases} 1 & \text{if } x = \frac{1}{2}, y \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise} \end{cases}$$

So the $x-$slice becomes,

$$f_x \equiv 0 \text{ for all } x \neq \frac{1}{2} \qquad \text{and} \qquad f_{\frac{1}{2}}(y) = \underbrace{\begin{cases} 1 & \text{if } y \in \mathbb{Q}^c \cap [0,1] \\ 0 & \text{if } y \in \mathbb{Q} \cap [0,1] \end{cases}}_{\text{Dirichlet Function}}$$

and $y-$slice,

$$\text{For } y \in \mathbb{Q} \cap [0,1],\ f_y(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$
$$\text{For } y \in \mathbb{Q}^c \cap [0,1],\ f_y \equiv 0$$

Clearly, $f_x \in \mathscr{R}(I)$ for all $x \in \frac{1}{2}$ but $f_{\frac{1}{2}} \notin \mathscr{R}(I)$ and $f_y \in \mathscr{R}(I)$ for every $y$. So,

$$\int_I f_y = 0 \implies \int_I \left( \int_I f_y(x)\, \mathrm{d}x \right) \mathrm{d}y = 0$$

But, $\int_I f_x$ doesn't exist for $x = \frac{1}{2}$ which means $x \longmapsto \int_I f_x$ is not a well-defined function on $[0,1]$. Hence, $\int_I \left( \int_I f_x(y)\, \mathrm{d}y \right) \mathrm{d}x$ doesn't exist. Yet $f \in \mathscr{R}(B^2)$. To see this, we fix $\epsilon > 0$ and consider the partition $P = P_1 \times P_2$ where,

$$\begin{cases} P_1 : 0 < \frac{1}{2} - \epsilon < \frac{1}{2} + \epsilon < 1 \\ P_2 : 0 < 1 \end{cases}$$

$$\text{So, } P = \left\{ \underbrace{\left[0, \frac{1}{2} - \epsilon\right] \times I}_{B_{\alpha_1}}, \underbrace{\left[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon\right] \times I}_{B_{\alpha_2}}, \underbrace{\left[\frac{1}{2} - \epsilon, 1\right] \times I}_{B_{\alpha_3}} \right\}.$$

Then $m_{\alpha_1} = m_{\alpha_2} = m_{\alpha_3} = 0$, $M_{\alpha_1} = M_{\alpha_3} = 0$ and $M_{\alpha_2} = 1$, which implies $U(f,P) - L(f,P) = 2\epsilon < 3\epsilon$. This shows that $f \in \mathscr{R}(B^2)$. Again $L(f,P) = 0$ for all $P \in \mathscr{P}(B^2)$ and hence,

$$\int_{B^2} f = 0$$

**Question.** Under which conditions does (17.3) hold?
*Answer.* Fubini's Theorem. The conditions for (17.3) to hold will be discussed in the next lecture.