Lecture 18

18.1 Fubini's Theorem

In this lecture, we explore Fubini's Theorem. Let's begin by setting up the necessary framework.

Consider the $(m+n)$-dimensional space, where m and n are positive integers. We can decompose a box $B^{m+n} \subseteq \mathbb{R}^{m+n}$ as the Cartesian product of two boxes, $B^{m+n} = B^m \times B^n$. Here, B^m represents a box in $\mathbb{R}^m$, and B^n represents a box in $\mathbb{R}^n$.

Now, suppose we have a partition $P \in \mathscr{P}(\mathbb{R}^{m+n})$. We can express this partition as the Cartesian product of two partitions, $P = P^m \times P^n$, where $P^m \in \mathscr{P}(\mathbb{R}^m)$ and $P^n \in \mathscr{P}(\mathbb{R}^n)$. The corresponding indexing set for this partition becomes $\Lambda(P) = \Lambda(P^m) \times \Lambda(P^n)$. Consequently, the elements of $\Lambda(P)$ can be written as $\alpha(P) = (\alpha(P^m), \alpha(P^n))$.

By extending this decomposition, we can also break down the elements of the boxes, $B_{\alpha(P)} = B_{\alpha(P^m)} \times B_{\alpha(P^n)}$.

Throughout this section, we take $x \in B^m$ and $y \in B^n$ to represent the point $(x, y) \in B^{m+n}$. For a bounded function $f \in \mathscr{B}(B^{m+n})$, we define the slice functions

- $f_x : B^n \to \mathbb{R}$ as $y \mapsto f(x, y)$ for all $x \in B^m$.
- $f_y : B^m \to \mathbb{R}$ as $x \mapsto f(x, y)$ for all $y \in B^n$.

It is worth noting that $f_x \in \mathscr{B}(B^n)$ and $f_y \in \mathscr{B}(B^m)$. For a fixed $x \in B^m$, we can compute the lower and upper integrals of f_x over B^n, denoted as $\underline{f}(x)$ and $\overline{f}(x)$ respectively. Similarly, we can compute the lower and upper integrals of f_y over B^m for fixed $y \in B^m$. These are given by,

$$\underline{f}(x) = \underline{\int}_{B^n} f_x(y)\ \mathrm{d}V(y) \quad \text{and} \quad \overline{f}(x) = \overline{\int}_{B^n} f_x(y)\ \mathrm{d}V(y)$$

with similar expressions for y. Now, let's state Fubini's Theorem.

Theorem 18.1.1 (Fubini's Theorem)

Let $f \in \mathscr{R}(B^{m+n})$. Then $\underline{f}, \overline{f} \in \mathscr{R}(B^n)$ and,

$$\int_{B^m} \underline{f} = \int_{B^m} \overline{f} = \int_{B^{m+n}} f$$

Consequently, we have the following corollaries,

Corollary

For any $f \in \mathscr{R}(B^{m+n})$, the following equalities hold,

$$\int_{B^m}\left(\underline{\int}_{B^n} f(x,y)\ \mathrm{d}V(y)\right)\mathrm{d}V(x) = \int_{B^m}\left(\overline{\int}_{B^n} f(x,y)\ \mathrm{d}V(x)\right)\mathrm{d}V(y)$$
$$= \int_{B^{m+n}} f(x,y)\ \mathrm{d}V(x,y)$$

Furthermore, if $f_x \in \mathscr{R}(B^n)$ for all x, then $\underline{f} = \overline{f}$ and

$$\int_{B^m}\left(\int_{B^n} f(x,y)\ \mathrm{d}V(y)\right)\mathrm{d}V(x) = \int_{B^{m+n}} f(x,y)\ \mathrm{d}V(x,y)$$

Corollary

If $f \in C(B^n)$, then all possible slice functions are continuous and hence Riemann Integrable. Thus, multidimensional integral becomes the iterated one-dimensional integrals,

$$\int_{B^n} f = \int\left(\int\cdots\int\left(\int f\ \mathrm{d}x_1\right)\mathrm{d}x_2\cdots\mathrm{d}x_{n-1}\right)\mathrm{d}x_n$$

where, x_i's can be in any order.

$(n=2)$ Hence, if $f \in C(B^2)$, then (17.3) holds,

$$\int_{B^2} f = \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f(x,y)\ \mathrm{d}y\right)\mathrm{d}x = \int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} f(x,y)\ \mathrm{d}x\right)\mathrm{d}y$$

Proof of the Fubini's Theorem. Let $P = P^m \times P^n$ be a partition of B^{m+n}. Then, we can express the lower sum $L(f,P)$ as follows,

$$L(f,P) = \sum_{\alpha(P)\in\Lambda(P)} m_{\alpha(P)}\,\mathrm{Vol}\left(B^{m+n}_{\alpha(P)}\right)$$
$$= \sum_{\alpha(P^m)\in\Lambda(P^m)} \underbrace{\sum_{\alpha(P^n)\in\Lambda(P^n)} m_{(\alpha(P^m),\alpha(P^n))}\,\mathrm{Vol}\left(B^n_{\alpha(P^n)}\right)}_{l_{P^m}}\mathrm{Vol}\left(B^m_{\alpha(P)^m}\right)$$

For each $x \in B^m$ and $\alpha(P^n) \in \Lambda(P^n)$, let $m_{\alpha(P^n)}(x) = \inf_{y\in B^n_{\alpha(P^n)}} f_x(y)$. It follows that $m_{\alpha(P^n)}(x) \geq m_{(\alpha(P^m),\alpha(P^n))}$ for every $x \in B^m_{\alpha(P^m)}$. Consequently, we have,

$$l_{P^m} \leq \sum_{\alpha(P^n)\in\Lambda(P^n)} m_{\alpha(P^n)}\,\mathrm{Vol}\left(B^n_{\alpha(P^n)}\right)$$
$$= L(f_x, P^n) \leq \underline{\int}_{B^n} f_x$$

Taking infimum over all $x \in B^m_{\alpha(P^m)}$, we obtain,

$$l_{P^m} \leq \inf_{x\in B^m_{\alpha(P^m)}} \underline{\int}_{B^n} f_x$$

$$= \inf_{x \in B^m_{\alpha(P^m)}} \underline{f}(x) = \underline{m}_{\alpha(P^m)}$$

Thus, the lower sum becomes,

$$L(f, P) \leq \sum_{\alpha(P^m) \in \Lambda(P^m)} \underline{m}_{\alpha(P^m)} \operatorname{Vol}\left(B^m_{\alpha(P^m)}\right) = L(\underline{f}, P^m)$$

Similarly, we can show that $U(f, P) \geq U(\underline{f}, P^m)$. Consequently, $\underline{f} \in \mathscr{R}(B^m)$, and we have,

$$\int_{B^m} \underline{f} \, \mathrm{d}V(x) = \int_{B^{m+n}} f$$

By following analogous arguments, we can show that $\overline{f} \in \mathscr{R}(B^m)$ and,

$$\int_{B^m} \overline{f} \, \mathrm{d}V(x) = \int_{B^{m+n}} f = \int_{B^m} \underline{f} \, \mathrm{d}V(x)$$

□

Question: Will the function be Riemann integrable if all the slices are Riemann integrable?
We will address this question later. In the meantime, let's conclude this lecture with an example.

Example 18.1.1

Consider the integral

$$\int_{[0,1]^2} xy \, \underbrace{\mathrm{d}x \, \mathrm{d}y}_{\mathrm{d}v}$$

We can evaluate this integral by iterated integration as follows,

$$\begin{aligned} \int_0^1 \left(\int_0^1 xy \, \mathrm{d}x \right) \mathrm{d}y &= \int_0^1 y \left(\int_0^1 x \, \mathrm{d}x \right) \mathrm{d}y \\ &= \int_0^1 \frac{y}{2} \, \mathrm{d}y = \frac{1}{4} \end{aligned}$$

Alternatively, we can also express it as, $\int_0^1 y \left(\int_0^1 x \, \mathrm{d}x \right) \mathrm{d}y = \left(\int_0^1 x \, \mathrm{d}x \right) \left(\int_0^1 y \, \mathrm{d}y \right)$, which yields the same result.