Lecture 19

19.1 Integration over Bounded Domain

Now that we know how to do integration over boxes, in this lecture we will discuss how to integrate a bounded function over an arbitrary bounded set.

Let $\Omega \subseteq \mathbb{R}^n$, and $f \in \mathscr{B}(\Omega)$ and Ω bounded, then there exists $B^n \supseteq \Omega$ where $B^n = \prod_{i=1}^n [a_i, b_i]$.

Definition 19.1.1

Given Ω bounded, let $B^n \supseteq \Omega$. For $f \in \mathscr{B}(\Omega)$, define

$$\tilde{f}_{B^n}(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in B^n \setminus \Omega \end{cases}$$

An immediate question that arises now is: If $B^n \supseteq \Omega$ and $\hat{B^n} \supseteq \Omega$ then will it be true that

(I) $\tilde{f}_{B^n} \in \mathscr{R}(B^n)$

(II) And if (I) holds, is it necessarily true that $\displaystyle\int_{B^n} \tilde{f}_{B^n} = \int_{\hat{B^n}} \tilde{f}_{\hat{B^n}}$.

It is in fact true that (I) $\implies$ (II), but we won't cover the proof here.

Definition 19.1.2

Let $f \in \mathscr{B}(\Omega)$. We say that $f \in \mathscr{R}(\Omega)$ if $\displaystyle\int_{B^n} \tilde{f}_{B^n}$ exists for some $B^n \supseteq \Omega$, and in this case we define

$$\int_\Omega f := \int_{B^n} \tilde{f}_{B^n}$$

Definition 19.1.3 ▶ Content Zero Sets

Let $S \subseteq \mathbb{R}^n$, we say that S is of content zero if for all $\varepsilon > 0$, there exists boxes $\{B_j^n\}_{j=1}^p$ (for some $p \in \mathbb{N}$) such that

$$S \subseteq \bigcup_{j=1}^p B_j^n \quad \text{and} \quad \sum_{j=1}^n \text{Vol}(B_j^n) < \varepsilon$$

For example a line segment in $\mathbb{R}^n$ is of content zero, provided $n > 1$. We then have the following theorems:

Theorem 19.1.1

(i) Let $f \in \mathscr{B}(B^n)$ and let $\mathcal{D} = \{x \in B^n \mid f \text{ is not continuous at } x\}$ be the set of discontinuities of f, if $\mathcal{D}$ is of content zero, then $f \in \mathscr{R}(B^n)$.

(ii) If S is a content zero set then $\operatorname{int}(S) = \emptyset$.

(iii) Let $\Omega \subseteq \mathbb{R}^n$ and $\mathcal{O}_n \subseteq \Omega$ is bounded. Let $f \in \mathscr{B}(\Omega)$ and $f|_{\mathcal{O}_n} \in C(\mathcal{O}_n)$, if $\bar{\Omega} \setminus \mathcal{O}_n$ is content zero then $f \in \mathscr{R}(\Omega)$ and

$$\int_\Omega f = \int_{\mathcal{O}_n} f$$

Particularly, (i) and (iii) of Theorem 19.1.1 are very important.

Now that we know how to integrate on arbitrary domains, the next question that comes to our mind is, does there exist a Fubini's theorem for integration over arbitrary sets? Before that we define elementary regions.

19.2 Two Elementary Regions

Definition 19.2.1 ▶ Elementary Regions

A set $\Omega \subseteq \mathbb{R}^2$ is y-simple/type I if there exists functions $\varphi_1, \varphi_2 \in \mathscr{B}([a, b])$ such that

$$\Omega = \{(x, y) \mid x \in [a, b],\ y \in [\varphi_1(x), \varphi_2(x)]\}$$

Similarly a set $\Omega \subseteq \mathbb{R}^2$ is x-simple/type II if there exists functions $\psi_1, \psi_2 \in \mathscr{B}([c, d])$ such that

$$\Omega = \{(x, y) \mid y \in [c, d],\ x \in [\psi_1(x), \psi_2(x)]\}$$

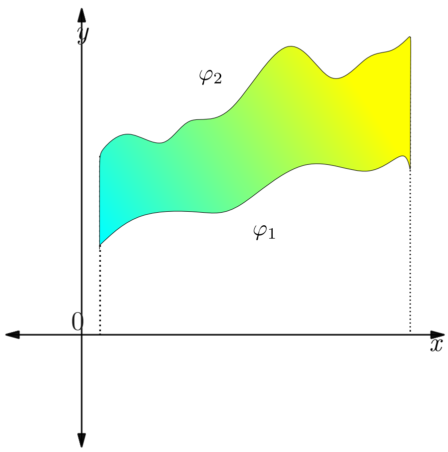

Figure 19.1: Example of a y-simple region.

Example 19.2.1 (Examples of elementary regions)

The region H given by

$$H = \{(x, y) \mid 0 \leq x \leq 1, \text{ and } x^2 \leq y \leq x\}$$

is a y-simple region. (see Figure 19.1)

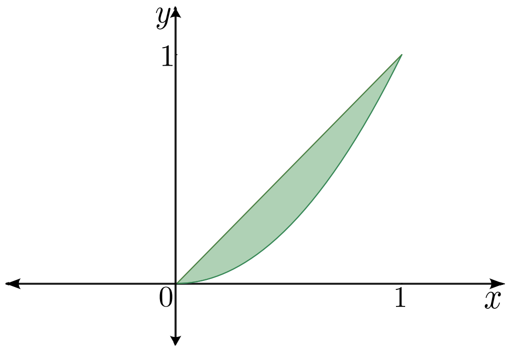

Figure 19.2: Plot of the region H.

Exercise. Show that the region bounded by $x^2 + y^2 \leq 1$ and $y \geq 0$ in $\mathbb{R}^2$ is an x-simple as well as a y-simple region.