Lecture 20

In the previous lecture we extended integration over boxes to over what we called elementary regions. This lecture explores the extention of Fubini's theorem to integration over such sets, and talk about some applications thereof. Towards the end, we discuss the celebrated change of variables formula of multivariable calculus.

20.1 Fubini's Theorem on Elementary Regions

For integration over elementary regions, Fubini's theorem takes the following form.

Theorem 20.1.1

Let $f \in \mathscr{R}(\Omega)$ where $\Omega \subseteq \mathbb{R}^2$ is a bounded elementary domain.

(1) If $\Omega = \{(x,y) \mid a \le x \le b, \text{ and } \varphi_1(x) \le y \le \varphi_2(x)\}$ and if $\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y$ exists for all $x \in [a,b]$ then

$$\int_\Omega f\,\mathrm{d}A = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$$

(2) Similarly we have

$$\int_\Omega f\,\mathrm{d}A = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y$$

when Ω is x-simple.

Proof. There exists $c, d \in \mathbb{R}$ such that $\Omega \subseteq [a,b] \times [c,d] = B^2$. We know $\tilde{f} \in \mathscr{R}(B^2)$ where

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in \Omega \\ 0 & \text{if } (x,y) \in B^2 \setminus \Omega \end{cases}$$

Since $\tilde{f} \in \mathscr{R}(B^2)$, and since $\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y$ exists for fixed x, hence

$$\tilde{f}(x,\cdot)\big|_{[\varphi_1(x),\varphi_2(x)]} \quad \text{and} \quad \tilde{f}(x,\cdot)\big|_{[c,d]\setminus[\varphi_1(x),\varphi_2(x)]} \equiv 0$$

are both Riemann integrable. Thus, we get that $\tilde{f}(x,\cdot)|_{[c,d]} \in \mathscr{R}([c,d])$ and hence $\int_c^d \tilde{f}(x,y)\,\mathrm{d}y$ exists for all $x \in [a,b]$ and further we have

$$\int_c^d \tilde{f}(x,y)\,\mathrm{d}y = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y \quad \forall x \in [a,b]$$

Then we get

$$\begin{aligned}
&\Longrightarrow \int_a^b \left(\int_c^d \tilde{f}(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x \\
&\overset{(*)}{\Longrightarrow} \int_{B^2} \tilde{f}\,\mathrm{d}A = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x \\
&\Longrightarrow \int_\Omega f\,\mathrm{d}A = \int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x
\end{aligned}$$

where $(*)$ follows from Theorem 18.1.1. An analogous argument works for the x-simple regions. □

Example 20.1.1

Let $f \in C(\Omega)$ where $\Omega = \left\{(x,y) \mid 0 \le x \le 1 - \frac{y}{2},\ \text{and } 0 \le y \le 2\right\}$. Then we can write Ω as a y-simple region as follows:

$$\Omega = \{(x,y) \mid 0 \le x \le 1,\ \text{and } 0 \le y \le 2(1-x)\}\,.$$

Now, using Theorem 20.1.1 we get the required result.

$$\int_\Omega f\,\mathrm{d}A = \int_0^2 \left(\int_0^{1-\frac{y}{2}} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y = \int_0^1 \left(\int_0^{2(1-x)} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x$$

Example 20.1.2

Let $B^2 = [0,\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, and we want to evaluate the integral $\displaystyle\int_{B^2} \sin(x+y)\,\mathrm{d}A$.

$$\begin{aligned}
\int_{B^2} \sin(x+y)\,\mathrm{d}A &= \int_{B^2} \sin x \cos y\,\mathrm{d}A + \int_{B^2} \sin y \cos x\,\mathrm{d}A \\
&= \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos y\,\mathrm{d}y\right)\left(\int_0^\pi \sin x\,\mathrm{d}x\right) + \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin y\,\mathrm{d}y\right)\left(\int_0^\pi \cos x\,\mathrm{d}x\right) \\
&= 4
\end{aligned}$$

Example 20.1.3

Let Ω be the region bounded by $y = 1$ and $y = x^2$, and we want to find $\displaystyle\int_\Omega x^2 y\,\mathrm{d}V$. We can write Ω as a y-simple region as follows:

$$\Omega = \{(x,y) \mid -1 \le x \le 1,\ \text{and } x^2 \le y \le 1\}$$

Then using Theorem 20.1.1 we get that

$$\begin{aligned}
\int_\Omega x^2 y\,\mathrm{d}A &= \int_{-1}^1 \left(\int_{x^2}^1 x^2 y\,\mathrm{d}y\right)\mathrm{d}x \\
&= \int_{-1}^1 x^2 \left(\frac{y^2}{2}\right)\Big|_{x^2}^1 \mathrm{d}x = \frac{2}{15}
\end{aligned}$$

Example 20.1.4

Compute $\int_{[0,1]^2} f \, \mathrm{d}A$ where $f : [0,1]^2 \to \mathbb{R}$ is given by

$$f(x,y) = \begin{cases} x & \text{if } y \leq x^2 \\ y & \text{if } y > x^2 \end{cases} \quad \forall \, (x,y) \in [0,1]^2$$

We define the regions

$$\Omega_1 = \{(x,y) \mid 0 \leq x \leq 1, \, 0 \leq y \leq x^2\} \quad \text{and} \quad \Omega_2 = \{(x,y) \mid 0 \leq x \leq 1, \, x^2 \leq y \leq 1\}.$$

Then $f|_{\Omega_1}$ and $f|_{\Omega_2}$ are both Riemann integrable, and while $f|_{y=x^2}$ is not continuous, the set $\{(x,x^2) \mid 0 \leq x \leq 1\}$ is of content zero. Hence f is integrable, and we can make sense of writing the given integral as a sum

$$\int_{[0,1]^2} f \, \mathrm{d}A = \int_{\Omega_1} f \, \mathrm{d}A + \int_{\Omega_2} f \, \mathrm{d}A$$

Using Theorem 20.1.1, we can simplify each of these parts:

$$\int_{\Omega_1} f \, \mathrm{d}A = \int_0^1 \left(\int_0^{x^2} x \, \mathrm{d}y \right) \mathrm{d}x = \frac{1}{4}$$

$$\int_{\Omega_2} f \, \mathrm{d}A = \int_0^1 \left(\int_{x^2}^1 y \, \mathrm{d}y \right) \mathrm{d}x = \frac{2}{5}$$

Thus, we have the result

$$\int_{[0,1]^2} f \, \mathrm{d}A = \frac{13}{20}$$

Example 20.1.5

Compute $\int_0^{\frac{\pi}{2}} \int_x^{\frac{\pi}{2}} \frac{\sin y}{y} \, \mathrm{d}y \, \mathrm{d}x$ using Fubini's theorem.

We consider the region $\Omega = \left\{(x,y) \mid 0 \leq x \leq \frac{\pi}{2}, \, x \leq y \leq \frac{\pi}{2}\right\} \subseteq \mathbb{R}^2$ can be written as a x-simple region as follows:

$$\Omega = \left\{(x,y) \mid 0 \leq y \leq \frac{\pi}{2}, \, 0 \leq x \leq y\right\}$$

This shows that

$$\begin{aligned} \int_\Omega \frac{\sin y}{y} \, \mathrm{d}A &= \int_0^{\frac{\pi}{2}} \int_x^{\frac{\pi}{2}} \frac{\sin y}{y} \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \int_0^{y} \frac{\sin y}{y} \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^{\frac{\pi}{2}} \sin y \, \mathrm{d}y \\ &= 1 \end{aligned}$$

20.2 Change of Variables

Before discussing the theorem in the multivariable case, we recall the change of variable rule for real valued functions on the real line.

Theorem 20.2.1 (Change of Variable on $\mathbb{R}$)

Let $\varphi : \mathcal{O}_1 \to \mathbb{R}$ be a C^1 function where $\varphi'(x) \neq 0$ for all $x \in \mathcal{O}_1$. Then, for $[a, b] \subseteq \mathcal{O}_1$ and $f \in C(\varphi[a, b])$, we have

$$\int_{\varphi(a)}^{\varphi(b)} f = \int_a^b (f \circ \varphi)\varphi'.$$

Here we effectively compensate for the change of variable by introducing the scale change factor of φ'. As we have seen, the scale change factor at a point for a transformation on $\mathbb{R}^n$ is given by the detereminant of the Jacobian matrix at that point. Thus, this theorem has the following natural extension to $\mathbb{R}^n$:

Theorem 20.2.2 (Change of Variable on $\mathbb{R}^n$)

Let $\varphi : \mathcal{O}_n \to \mathbb{R}^n$ be an injective and C^1 function, where $\det(J_\varphi(x)) \neq 0$ for all $x \in \mathcal{O}_n$. Let $\Omega \subseteq \mathcal{O}_n$, then for $f \in \mathscr{R}(\varphi(\Omega))$

$$\int_{\varphi(\Omega)} f \, \mathrm{d}V = \int_\Omega (f \circ \varphi)|\det J_\varphi|$$

Although it is not too hard to get a feel for the theorem from its applications, the proof is quite long and technical, and thus omitted. We recommend the interested and daring readers to have a look at page 67 of *Calculus on Manifolds* by Michael Spivak. In the next lecture, we will discuss some applications of this result.