Lecture 22

22.1 Curves and Surfaces

We now study the basic concepts of curves and surfaces as subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$ (mainly) with a given parametrization, but also as subsets defined by equations. The connection from equations to parametrization is drawn by means of the Implicit function theorem.

Definition 22.1.1 ▶ Parametrized Curve

A **parametrized curve** is a continuous function $\gamma : [a, b] \to \mathbb{R}^n$. We say that parametrized curve is C^1 if $t \mapsto \gamma_i(t)$ is C^1 for all $i = 1, \ldots, n$. A parametrized curve $\gamma : I \to \mathbb{R}^n$ is **smooth** if $\gamma'(t) \neq \mathbf{0}$ for all $t \in I$. The **path** of a parametrized curve γ is the set

$$\{\gamma(t) \mid t \in [a, b]\}$$

Let us consider some examples.

Example 22.1.1

1. Let $\gamma : [0, 1] \to \mathbb{R}^2$ be $\gamma(t) = (1 - 2t, 2 + t)$. Clearly γ is C^1 and $\gamma'(t) = (-2, 1) \neq \mathbf{0}$ and thus γ is smooth.
2. $\gamma : [0, 2\pi] \to \mathbb{R}^2$ given by $\gamma : t \mapsto (r \cos t, r \sin t)$ where $r > 0$ is constant. Then $\gamma'(t) = (-r \sin t, r \cos t) \neq \mathbf{0} \, \forall \, t \in [0, 2\pi]$, thus γ is smooth.
3. Fix $r > 0$ and $c \neq 0$, and define

$$\begin{aligned} \gamma : [0, n\pi] &\to \mathbb{R}^3 \\ t &\mapsto (r \cos t, r \sin t, ct) \end{aligned}$$

   Then $\gamma'(t) = (-r \sin t, r \cos t, c) \neq \mathbf{0}$ and hence γ is smooth. The path of γ is called a **helix**.
4. $\gamma : [-1, 1] \to \mathbb{R}^2$ given by $\gamma(t) = (|t|, t)$, then γ is not C^1, hence it is not smooth.
5. $\gamma : [0, 1] \to \mathbb{R}^2$ given by $\gamma(t) = (0, t^2)$, then even though γ is C^1, but it is not smooth as $\gamma'(0) = \mathbf{0}$.
6. $\gamma : [0, 2\pi] \to \mathbb{R}^2$ given by $\gamma : t \mapsto (r \cos t, r \sin t)$, then path of γ is given by

$$\begin{aligned} \{(r \cos t, r \sin t) \mid t \in [0, 2\pi]\} &= \{(x, y) \mid x^2 + y^2 = r^2\} \\ &= \text{path of } \tilde{\gamma} \end{aligned}$$

   where $\tilde{\gamma}(t) = (r \cos 2t, r \sin 2t)$.

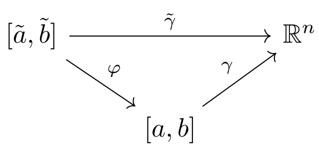
Definition 22.1.2 ▶ Piecewise Smooth Curve

A parametrized curve $\gamma : [a, b] \to \mathbb{R}^n$ is called piecewise smooth if there exists a partition $a = t_0 < t_1 < \cdots < t_m = b$ such that

$$\gamma|_{[t_{i-1}, t_i]} \text{ is smooth } \forall\, i \in [m]$$

Definition 22.1.3 ▶ Equivalent Curves

Two parametrized curves $\gamma : [a, b] \to \mathbb{R}^n$ and $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{R}^n$ are equivalent, denoted by $\gamma \sim \tilde{\gamma}$ if there exists a strictly increasing surjective function which is differentiable (even C^1), $\varphi : [\tilde{a}, \tilde{b}] \to [a, b]$ such that $\tilde{\gamma} = \gamma \circ \varphi$.



Definition 22.1.4

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a C^1 curve, then

(i) $\|\gamma'(t)\| :=$ speed of γ at time t.

(ii) $\displaystyle\int_a^b \|\gamma'(t)\| \mathrm{d}t :=$ arc length of γ.

Let's try to look at more natural how equation (ii) in the previous definition gives us the arc length of a curve γ.

Let $\gamma : [a, b] \to \mathbb{R}^n$ and let $\mathcal{P} := a = t_0 < t_1 < \cdots < t_m = b$ be a partition of the interval $[a, b]$. Now define

$$\ell(\gamma, \mathcal{P}) = \sum_{i=1}^{m} \|\gamma(t_{i-1}) - \gamma(t_i)\|$$

Definition 22.1.5

A curve $\gamma : [a, b] \to \mathbb{R}^n$ is **rectifiable** or said to have arc length if

$$\lim_{\substack{\|\mathcal{P}\| \to 0 \\ \mathcal{P} \in \mathscr{P}[a,b]}} \ell(\gamma, \mathcal{P}) = \ell(\gamma) \text{ exists}$$

which is equivalent to saying that for all $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\|\ell(\gamma, \mathcal{P}) - \ell(\gamma)\| < \varepsilon \quad \forall \mathcal{P} \in \mathscr{P}([a, b]) \text{ such that } \|\mathcal{P}\| < \delta$$

Theorem 22.1.1

For a piecewise smooth curve $\gamma : [a, b] \to \mathbb{R}^n$ it is rectifiable and $\ell(\gamma) = \displaystyle\int_a^b \|\gamma'(t)\| \mathrm{d}t$.

Remark. Rectifiable curve $\not\Rightarrow$ piecewise smooth, counter example: Cantor's function (popularly called the Devil's staircase).

Theorem 22.1.2

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a rectifiable parametrized curve and let $\tilde{\gamma} = \gamma \circ \varphi$, where φ is a strictly increasing surjective and continuous function, then $\tilde{\gamma}$ is rectifiable and $\ell(\gamma) = \ell(\tilde{\gamma})$.

Theorem 22.1.3

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a smooth curve, then there exists a parametrization φ such that $\|\tilde{\gamma}'(s)\| = 1$ for all $s \in [c, d]$.