Lecture 23

We will begin this lecture with few examples.

Example 23.0.1 (Path of a Projectile)

$$\begin{aligned}\gamma(t) &= (\alpha t, \beta t - 16t^2)\\ \implies \gamma'(t) &= (\alpha, \beta - 32t)\\ \text{path length} &= \int \|\gamma'(t)\|\,\mathrm{d}t\\ &= \int \sqrt{\alpha^2 + (\beta - 32t)^2}\,\mathrm{d}t\end{aligned}$$

Example 23.0.2 (Perimeter of a Circle)

Parametrization of a circle of radius r is given by $\gamma(t) = (r\cos(t), r\sin(t)), t \in [0, 2\pi)$.

$$\begin{aligned}\gamma'(t) &= (-r\sin(t), r\cos(t))\\ \ell(\gamma) &= \int_0^{2\pi} \|\gamma'(t)\|\mathrm{d}t\\ &= \int_0^{2\pi} r\,\mathrm{d}t\\ &= 2\pi r\end{aligned}$$

Example 23.0.3 (Arc Length of graph of functions)

Let, $f : [a, b] \to \mathbb{R}$ be a C^1 function. Consider $\gamma(t) = (t, f(t))$. It is a smooth curve.

$$\begin{aligned}\gamma'(t) &= (1, f'(t))\\ \ell(\gamma) &= \int_a^b \|\gamma'(t)\|\mathrm{d}t\\ &= \int_a^b \sqrt{1 + f'(t)^2}\,\mathrm{d}t\end{aligned}$$

23.1 Line Integrals

To integrate a function over a curve we use **Line integral**. The function we should integrate maybe a **Scalar Field** or a **Vector Field**.(A quick example of a Vector Field: $f : \mathcal{O}_n \to \mathbb{R}$ be a differentiable function, then $\boldsymbol{\nabla} f$ is a vector field.)

Question. Given a scalar field $f : \mathcal{O}_n \to \mathbb{R}$ and $\gamma \equiv \mathcal{C}$ be a curve, we want to define $\int_{\mathcal{C}} f$. But exactly how we can do this?

Answer. $\mathcal{C}$ is a curve, so it is bounded subset of $\mathbb{R}^n$. How about thinking of **Riemann Integration**? For $n \geq 2$, $\mathcal{C}$ is **content zero** in $\mathbb{R}^n$. This does not make any sense! The right way is as following.

Let, $\gamma : [a, b] \to \mathbb{R}^n$ be a **smooth curve** (or piecewise smooth) and $\mathcal{C} := \operatorname{ran}(\gamma)$ (on other words path of γ). Let, $f \in \mathscr{B}(\mathcal{C})$. Given $\mathcal{P} \in \mathscr{P}[a, b]$, $\mathcal{P} : a = t_0 < t_1 < \cdots < t_m = b$.

Let, $I_i = [t_{i-1}, t_i]$ be the sub-intervals and $\mathcal{C}_i = \gamma(I_i)$. Since, γ is smooth there is nice correspondence between I_i and $\mathcal{C}_i$. Also denote s_i by $\|\gamma(t_i) - \gamma(t_i)\|$. As previous, define $m_i = \inf_{\mathcal{C}_i} f$ and $M_i = \sup_{\mathcal{C}_i} f$.

$$U(f, \mathcal{P}) = \sum_{i=1}^{m} M_i \cdot s_i$$

$$L(f, \mathcal{P}) = \sum_{i=1}^{m} m_i \cdot s_i$$

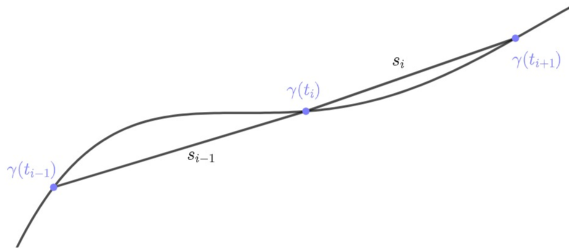

Figure 23.1: Curve $\mathcal{C}$

The above expressions are same as upper and lower Riemann sum respectively. This opens up "The Pandora's box!".

We can now use all the theory we used for the standard Riemann Integrals. We say f is line integrable over γ if,

$$\inf_{\mathcal{P} \in \mathscr{P}[a,b]} U(f, \mathcal{P}) = \sup_{\mathcal{P} \in \mathscr{P}[a,b]} L(f, \mathcal{P})$$

More over we will write the common value of the above equality as $\int_{\mathcal{C}} f$ and call this "The Line Integral over curve $\mathcal{C}$".

Now we can invoke all theory we derived for 1 variable integration! We call $\mathscr{R}(\mathcal{C})$ the set of all Riemann integrable functions over $\mathcal{C}$.

Theorem 23.1.1

Let, γ be a "Rectifiable" smooth(or piecewise smooth) and $\mathcal{C} = \operatorname{ran}(\gamma)$ and $f \in \mathscr{B}(\mathcal{C})$. Then,

1. $f \in C^0(\mathcal{C}) \implies f \in \mathscr{R}(\mathcal{C})$
2. $$f \in \mathscr{R}(\mathcal{C}) \iff \lim_{||\mathcal{P}|| \to 0} \sum_{i=1}^{m} f(\zeta_i) s_i \quad \text{exist and equal to} \int_{\mathcal{C}} f.$$
   Here, ζ_i is tag of the interval I_i.
3. (This requires smoothness) If γ is C^1 and smooth, $f \in \mathscr{R}(\mathcal{C})$, then
   $$\int_{\mathcal{C}} f = \int_a^b f(\gamma(t)) \|\gamma'(t)\| \, \mathrm{d}t$$

Proof. **Exercise.**

In the above theorem equation in 3 is also independent of choice of Parametrization of γ. Because any other smooth parametrized curve $\tilde{\gamma} = \gamma \circ \varphi$ where, φ is an onto continuous function.

Facts: $\mathcal{C}$ be a piecewise smooth, parametrized curve γ. $f, g \in \mathscr{R}(\mathcal{C})$ and $r \in \mathbb{R}$,then

- $\displaystyle\int f + rg = \int f + r\int g$
- $f \geq g$ over $\mathcal{C}$ then $\displaystyle\int f \geq \int g$
- $\displaystyle\int |f| \geq \left|\int f\right|$
- If $a < d < b$, if $\gamma_1 := \gamma|_{[a,d]}$ and $\gamma_2 := \gamma|_{[d,b]}$ then
$$\int_{\mathcal{C}} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

We have resolved the problems for Scalar field. **What about vector fields?**

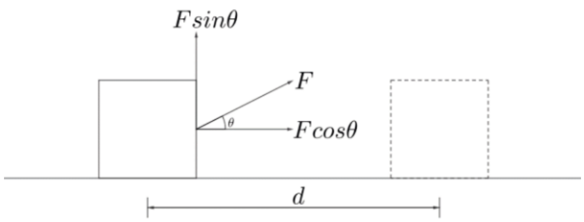

Figure 23.2: Work done by a constant Force

Suppose a particle moves a distance d under a constant force F, then work done by the force is $Fd\cos\theta = \vec{F} \cdot \vec{d}$.
If the force was not constant throughout the path then how can we calculate work done by that force? Consider the case where F is the vector field (Force in this case) defined over a curve (path)γ. Here, $\gamma : [a, b] \to \mathbb{R}^n$ and $\mathcal{C} = \text{ran}(\gamma)$. So, work done throughout the whole path will be,
$$\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r}$$
Which is equal to, $\displaystyle\int_a^b \vec{F}(\gamma(t)) \cdot \boldsymbol{\nabla}\gamma(t)\,\mathrm{d}t$
Now we will look into some examples.

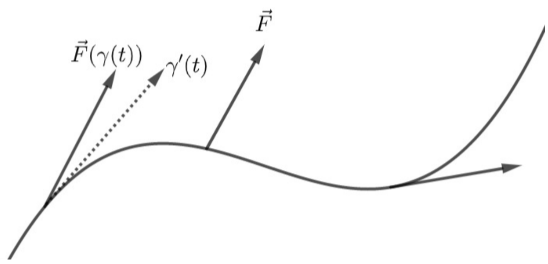

Figure 23.3: $\vec{F}$ is the vector field over the curve γ

Example 23.1.1

Find work done by the field $F(x, y, z) = (xy, xz, yz)$ along the curve $\gamma(t) = (t^2, -t^3, t^4), t \in [0, 1]$.
Answer.

$$\gamma'(t) = (2t, -3t^2, 4t^3)$$

$$\implies \text{Work done} = \int_0^1 (-t^5, t^6, -t^7) \cdot (2t, -3t^2, 4t^3)\, \mathrm{d}t$$
$$= -\frac{31}{88}$$

23.1.1 Line Integration of a Vector Field

$F : \mathcal{O}_n \to \mathbb{R}^n$ be a vector field and $\gamma : [a, b] \to \mathcal{O}_n$ be a curve. We consider a partition $\mathcal{P} : a = t_0 < t_1 < \cdots < t_m = b$, Let $\mathcal{C}_i = \gamma|_{[t_{i-1}, t_i]}$ and $\gamma_i = \gamma(t_i)$, $\Delta r_i = \gamma_i - \gamma_{i-1}$.

$$R(F; \mathcal{P}) = \sum_{i=1}^{m} F(\gamma_i) \cdot \Delta r_i$$

$$\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r} = \lim_{\|\mathcal{P}\| \to 0} R(F, \mathcal{P}) \qquad \text{(if the limit exists)}$$

Just like the scalar field, if γ is C^1 and smooth, then

$$\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r} = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, \mathrm{d}t \tag{23.1}$$