Lecture 25

25.1 Tangent Plane Of $\mathcal{G}(f)$

Let, $f : \mathcal{O}_2 \to \mathbb{R}$ be a C^1 function and $r(u,v) = (u, v, f(u,v))$. Here $\operatorname{ran}(r)$ defines a surface $\mathcal{G}(f)$ as we have showed in Example 24.3.1. We also have calculated $r_v \times r_u = (-f_u, -f_v, 1)$. Now using (24.1) we can write down the equation of **Tangent space** at a point $P = (a, b, f(a,b))$ on $\mathcal{G}(f)$. Equation of the tangent space $T_P\mathcal{S}$,

$$\begin{aligned} & f_u(a,b)(x-a) + f_v(a,b)(y-b) - (z - f(a,b)) = 0 \\ \implies & z = f(a,b) + f_u(a,b)(x-a) + f_v(a,b)(y-b) \end{aligned} \tag{25.1}$$

Equation of **Normal** at the point P on the surface $\mathcal{G}(f)$ is,

$$\frac{x-a}{-f_u(a,b)} = \frac{y-b}{-f_v(a,b)} = \frac{z-f(a,b)}{1} \tag{25.2}$$

Example 25.1.1 (Equation of Tangent and Normal to $z = f(x,y) = \frac{2x}{y} - y^2$ at $(1,1,1)$)

Solution. This is a graph function so obviously a surface $f_x(x,y) = \frac{2}{y}$ and $f_y = -\frac{2x}{y^2} - 2y$. So,$\langle -f_x, -f_y, 1\rangle = \langle -2, 4, 1\rangle$. So equation of Normal is $\frac{x-1}{-2} = \frac{y-1}{4} = \frac{z-1}{1}$ and the equation of Tangent Plane is, $2(x-1) - 4(y-1) - (z-1) = 0$. ■

Example 25.1.2 (Use Tangent Plane to approximate $(1.99)^2 - \frac{1.99}{1.01}$)

Solution. Consider $z = x^2 - \frac{x}{y} = f(x,y)$. This describes a surface. Now consider $P = (2,1,2)$ be the point on the surface. Here,$\langle -f_x, -f_y, 1\rangle = \langle -3, -2, 1\rangle$. So, equation of Tangent plane at P is,

$$z = 2 + 3(x-2) + 2(y-1)$$

The given expression can be approximated as, (by putting value of x, y in the above equation of tangent plane) $z(1.99, 1.01) \approx 1.99$. ■

Our next goal is to calculate area of different surfaces. We will start with very basic example, that is, area of a plane.

25.2 Surface Area

Suppose P_0, P_1, P_2 be the points on $\mathbb{R}^3$ and coordinate vector of the points is given by,

$$\overrightarrow{OP_0} = \langle a_0, b_0, c_0 \rangle$$
$$\overrightarrow{OP_1} = \langle a_1, b_1, c_1 \rangle$$
$$\overrightarrow{OP_2} = \langle a_2, b_2, c_2 \rangle$$

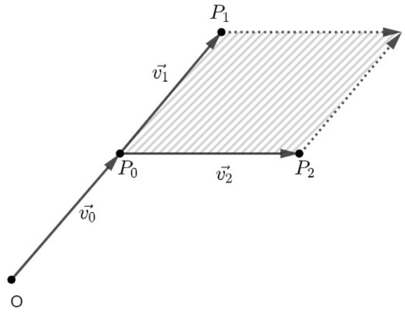

Figure 25.1: Plane $\mathcal{S}$

We will actually look at the parallelogram generated by,

$$\vec{v_1} = \overrightarrow{P_0P_1}$$
$$\vec{v_2} = \overrightarrow{P_0P_2}$$

Any point inside the parallelogram must look like $\vec{v_0} + t_1\vec{v_1} + t_2\vec{v_2}$ for some $t_1, t_2 \in [0, 1]$. So the parallelogram can be explicitly written as,

$$\mathcal{S} = \{\vec{v_0} + t_1\vec{v_1} + t_2\vec{v_2} \mid 0 \leq t_1, t_2 \leq 1\}$$

We know area of $\mathcal{S}$ is $\|\vec{v_1} \times \vec{v_2}\|$. We can describe this plane differently. If the equation of the plane was $z = ax+by+c$, then the surface of the plane can be described by $\mathcal{S} = \{(x, y, ax+by+c) \mid (x, y) \in B^2\}$. Area of $\mathcal{S} = \sqrt{1 + a^2 + b^2} \times \text{Area}(B^2)$. Now we should move forward to the general case.
Let, $r : B^2 \to \mathbb{R}^3$ be a function defined as $r(x, y) = (x, y, f(x, y))$ (Here f is C^1 function). Let, $\text{ran}(r)$ be the surface $\mathcal{S}$.

As we have done in the case of Riemann Integration. We should make partition of B^2 into tiny boxes. Let, $\mathcal{P} \in \mathscr{P}(B^2)$. Then,

$$B = \bigcup_{\alpha \in \Lambda(\mathcal{P})} B^2_\alpha$$

For any $\alpha \in \Lambda(\mathcal{P})$ fix $(x_\alpha, y_\alpha) \in B^2_\alpha$. Consider the tangent plane of $\mathcal{S}$ at $r(x_\alpha, y_\alpha)$ over B^2_α. Now the Tangent Plane at $r(x_\alpha, y_\alpha)$ is given by,

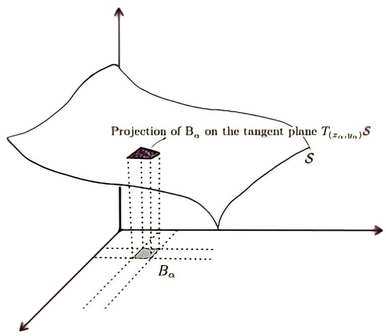

Figure 25.2: $T_{r(x_\alpha,y_\alpha)}\mathcal{S}$

$$\begin{aligned} z - f((x_\alpha, y_\alpha)) &= (Df)(x_\alpha, y_\alpha) \cdot (x_\alpha, y_\alpha) \\ \implies z &= (Df)(x_\alpha, y_\alpha) + f(x_\alpha, y_\alpha) \\ \implies z &= f_x(x_\alpha, y_\alpha) \cdot (x - x_\alpha) + f_y(x_\alpha, y_\alpha) \cdot (y - y_\alpha) + f(x_\alpha, y_\alpha) \end{aligned}$$

So, Area of $T_{r(x_\alpha,y_\alpha)}\mathcal{S}$ over B^2_α is

$$\sqrt{1 + f_x^2(x_\alpha, y_\alpha) + f_y^2(x_\alpha, y_\alpha)} \times \text{Area}(B_\alpha)^2$$

$$= \|r_x \times r_y\| \times \text{Area}(B_\alpha)^2$$

Since f is C^1 function so is r. So the total area of the surface $\mathcal{S}$ is given by,

$$\begin{aligned}\text{Area}(\mathcal{S}) &= \lim_{\|\mathcal{P}\|\to 0} \|r_x \times r_y\| \times \text{Area}(B_\alpha)^2 \\ &= \int_{B^2} \|r_x \times r_y\| \,\mathrm{d}A \\ &= \int_{B^2} \sqrt{1 + f_x^2 + f_y^2}\,\mathrm{d}A \quad \text{(In this case)}\end{aligned}$$

Will will do the above integration over any bounded set Ω as we have done in Riemann integration chapter. Over a bounded set Ω area of $\mathcal{S}$ is given by,

$$\text{Area}(\mathcal{S}) = \int_{\Omega} \sqrt{1 + f_x^2 + f_y^2}\,\mathrm{d}A \tag{25.3}$$

The General method for finding surface area is described by the following theorem.

Theorem 25.2.1

Let $\mathcal{R} \subseteq \mathbb{R}^2$ be a region. $r : \mathcal{R} \to \mathbb{R}^3$ be the parametrization of the surface $\mathcal{S}$. Then,

$$\text{Area}(\mathcal{S}) = \int_{\mathcal{R}} \|r_u \times r_v\| \,\mathrm{d}A$$