Lecture 27

27.1 Conservative Vector Fields

In the previous lecture we introduced the notion of an oriented surface. For an oriented surface $S \subseteq \mathbb{R}^3$, we call the orientation vector field $\vec{n} : S \to \mathbb{R}^3$ the **normal vector field**. Now we give an example of such a vector field.

Example 27.1.1

Take $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$.
Then

$$\vec{n}_1(x) = x \;\forall x \in \mathbb{S}^{n-1}$$

and

$$\vec{n}_2(x) = -x \;\forall x \in \mathbb{S}^{n-1}$$

are the two normal vector fields on the sphere.

Generally we consider the outward normal vector, i.e., the normal vector field given by $\vec{n}_1$ as the standard normal vector field on the sphere.

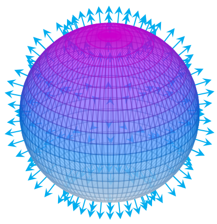

Figure 27.1: Standard normal vector field on a sphere

Formula for Normal Vector Field.
Let $\mathcal{G}(f) = \{(x, y, f(x, y)) \mid (x, y) \in \mathcal{O}_2\}$ where $f : \mathcal{O}_2 \to \mathbb{R}$ is a C^1 function. Then a parametrization of the surface $\mathcal{G}(f)$ is given by the function

$$\vec{r} : \mathcal{O}_2 \to \mathbb{R}^3$$
$$(x, y) \mapsto (x, y, f(x, y))$$

Then we have

$$\vec{r}_x \times \vec{r}_y = (-f_x, -f_y, 1)$$

Then a normal vector field is given by

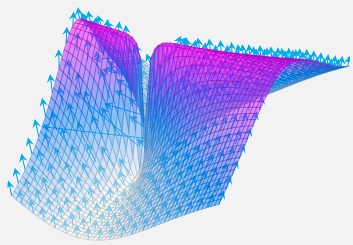

Figure 27.2: Normal vector fields on a graph surface

$$\vec{n}(x, y) = \frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|}$$

Unless otherwise mentioned this will be our standard orientation of the normal vector field.

Usually computation of $\int_S \vec{F} \cdot \mathrm{d}\vec{S}$ is complicated, let us look at some examples to gain more familiarity.

Example 27.1.2

Consider the vector field $\vec{F}(x,y,z) = (x,y,z)$ on $S = \operatorname{ran}(r)$, where

$$\vec{r}(x,y) = (\cos x, \sin x, y) \quad 0 \le x \le \frac{\pi}{2}, 0 \le y \le 1$$

Then $\vec{r}_x \times \vec{r}_y = (\cos x, \sin x, 0)$, so $\vec{n}(x,y) = (\cos x, \sin x, 0)$ is a normal vector field.

$$\begin{aligned}
\int_S \vec{F} \cdot \mathrm{d}\vec{S} &= \int_S \vec{F} \cdot \vec{n} \mathrm{d}s \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \vec{F}(\vec{r}(x,y)) \cdot (\vec{r}_x \times \vec{r}_y) \, \mathrm{d}A \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} (\cos x, \sin x, y) \cdot (\cos x, \sin x, 0) \mathrm{d}A \\
&= \int_0^1 \int_0^{\frac{\pi}{2}} \mathrm{d}A \\
&= \frac{\pi}{2}
\end{aligned}$$

We already know that $\int_{\mathcal{C}} \boldsymbol{\nabla} f \cdot \mathrm{d}r = f(B) - f(A)$, now a natural question that arises is

Question: Given $\vec{F}$, does there exist f a scalar field such that $\boldsymbol{\nabla} f = \vec{F}$?

Definition 27.1.1 ▶ Conservative Vector Field

A vector field $\vec{F}$ on $\mathcal{O}_n$ is called **conservative** if there exists a scalar field $f \in C^1(\mathcal{O}_n)$ such that $\boldsymbol{\nabla} f = \vec{F}$, then f is called the **potential function**.

Theorem 27.1.1

Let $\vec{F}$ be a vector field over $\mathcal{O}_n$, the following are equivalent:

1. $\vec{F}$ is conservative.
2. $\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}r = 0$, for all closed and piecewise smooth curve $\mathcal{C}$.
3. $\int_{\mathcal{C}_1} \vec{F} \cdot \mathrm{d}r = \int_{\mathcal{C}_2} \vec{F} \cdot \mathrm{d}r$, for all curves $\mathcal{C}_1$ and $\mathcal{C}_2$ with same initial and end points.

Question: Given a vector field $\vec{F}$, can we conclude $\vec{F}$ is conservative? (NO!)

We will give a general picture for the most common case, when $n = 3$. Let $\vec{F} = (P,Q,R)$ where P,Q,R are scalar fields. Now if $\vec{F} = \boldsymbol{\nabla} f$ for some scalar field f, then we would have

$$\begin{aligned}
f_x &\equiv \frac{\partial f}{\partial x} = P \\
f_y &\equiv \frac{\partial f}{\partial y} = Q \\
f_z &\equiv \frac{\partial f}{\partial z} = R
\end{aligned} \tag{27.1}$$

Then we can define **curl** of a vector field

$$\nabla \times \vec{F} := \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Then expanding this out and using the relations (27.1) and others we get that $\nabla \times \vec{F} = 0$. So, we have proved that if $\vec{F}$ is conservative then $\nabla \times \vec{F} = 0$.

Remark. Thus, a necessary condition for a vector field to be conservative is that, its curl should be the zero vector field.

Example 27.1.3

Let $\vec{F}(x, y) = (y - 3, x + 2) = (P, Q)$ (say), then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 1$. Let f be a possible potential function, then

$$\frac{\partial f}{\partial x} = y - 3 \quad \text{and} \quad \frac{\partial f}{\partial y} = x + 2$$

Then by **Fundamental Theorem of Calculus** (assuming domain is convex) we get

$$f(x, y) = xy - 3x + g(y)$$

But then using $\frac{\partial f}{\partial y} = x + 2$ we get

$$x + g'(y) = \frac{\partial f}{\partial y} = x + 2 \Rightarrow g'(y) = 2$$

Therefore taking $f(x, y) = xy - 3x + 2y$ gives us a potential function for the vector field $\vec{F}$.

Remark. This approach works for all $\vec{F}$ such that $\nabla \times \vec{F} = 0$ and the domain is convex.

Example 27.1.4

Let $\vec{F}(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) = (P, Q)$ (say) on $\mathbb{R}^2 \setminus \{0\}$. Then we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, but we will show that $\vec{F}$ is not conservative. Consider the curve

$$\mathcal{C} : \gamma(t) = (\cos t, \sin t), \quad 0 \leq t \leq 2\pi$$

then

$$\begin{aligned} \int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}r &= \int_0^{2\pi} \vec{F}(\gamma(t)) \cdot \gamma'(t)\, \mathrm{d}t \\ &= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, \mathrm{d}t \\ &= \int_0^{2\pi} \mathrm{d}t \\ &= 2\pi \end{aligned}$$

But $\mathcal{C}$ is clearly a closed curve, hence by Theorem 27.1.1 we must have $\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}r = 0$. (Contradiction!)

27.2 Green's Theorem

Definition 27.2.1 ▶ Simply Connected Domain

Let $\mathcal{D}$ be an open and connected set. Let $\mathcal{C}$ be a simple and closed curve if $\mathcal{C}$ can be shrunk continuously to a point inside $\mathcal{D}$, then we say $\mathcal{D}$ is **simply connected**.

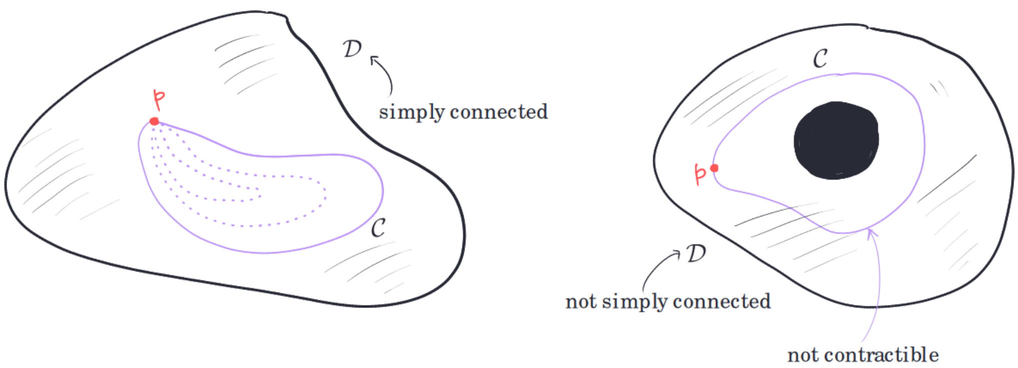

Figure 27.3: Examples of simply connected and not simply connected region

Theorem 27.2.1 (Green's Theorem)

Let $\mathcal{R} \subseteq \mathbb{R}^2$ be a simply connected domain with boundary curve $\mathcal{C}$ where parametrization is taken in anti-clockwise direction. Let $\vec{F} = (P, Q)$ be a C^1 vector field on $\mathcal{R}$, then

$$\int_{\mathcal{C}} \vec{F} \cdot \mathrm{d}r := \int_{\mathcal{C}} P \,\mathrm{d}x + Q \,\mathrm{d}y = \int_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A$$

What happens when $\mathcal{R}$ is not simply connected?

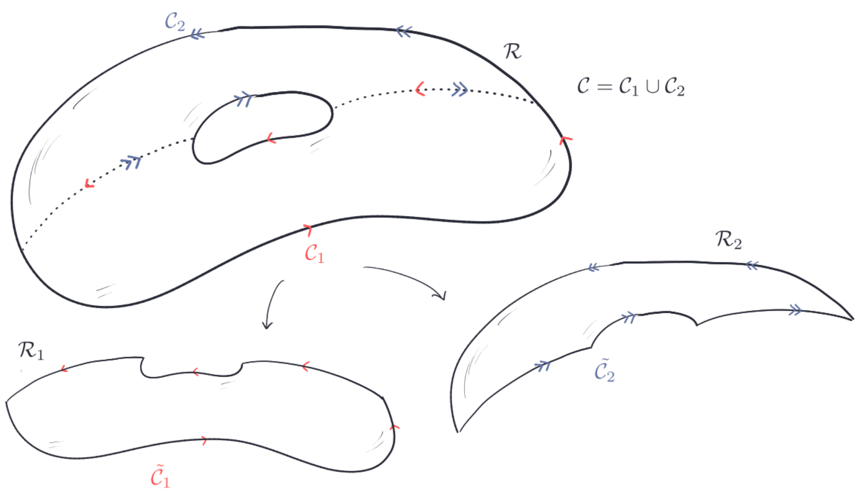

Figure 27.4: You break up the region $\mathcal{C}$ with the hole into two regions without holes $\mathcal{C}_1$ and $\mathcal{C}_2$.

$$\begin{aligned}
\int_{\mathcal{C}} P\,\mathrm{d}x + Q\,\mathrm{d}y &= \int_{\tilde{\mathcal{C}}_1} P\,\mathrm{d}x + Q\,\mathrm{d}y + \int_{\tilde{\mathcal{C}}_2} P\,\mathrm{d}x + Q\,\mathrm{d}y \\
&= \int_{\mathcal{R}_1} (Q_x - P_y)\,\mathrm{d}A + \int_{\mathcal{R}_2} (Q_x - P_y)\,\mathrm{d}A \\
&= \int_{\mathcal{R}} (Q_x - P_y)\,\mathrm{d}A
\end{aligned}$$