# Special lecture -2

Recall the Bloch's theorem. We will look at few corollary of this theorem.

COROLLARY. *If $f$ is holomorphic in domain $G$, $f'(c) \neq 0$ means $f(G)$ contains a disc of radii*

$$\frac{1}{12} s|f'(c)|$$

## Towards little Picard

> Lemma 14.0.1
>
> Let, $G \subseteq \mathbb{C}$ simply-connected domain, let $f \in \mathbf{Hol}(G)$ and it's image con't contain $-1, 1$, then $f = \cos F$ for some holomorphic $f$.

**Proposition** Let $G \subseteq \mathbb{C}$ be a simply connected domain, $f(G)$ don't contain $0, 1$ then there exist $g \in \mathbf{Hol}(G)$ such that,

$$f(z) = \frac{1 + \cos(\cos \pi g)}{2}$$

and image of $G$ does not contain any disc of radius 1.

If we take $f$ to be as the thing given above, since $f$ misses $0, 1$, $2f - 1$ will miss $-1, 1$. Now consider a set,

$$\mathbf{A} = \left\{ m \pm i \frac{\log\left(n + \sqrt{n^2 - 2}\right)}{\pi} : n \geq 2, m \in Z \right\}$$

This set will intersect disc of radius 1. But we will show $g(G) \cap \mathbf{A} = \emptyset$. Take $a \in \mathbf{A}$ and $a = m \pm i \frac{\log\left(n+\sqrt{n^2-2}\right)}{\pi}$ so we will have

$$\cos \pi a = (-1)^m n$$
$$f(a) = \frac{1 + (-1)^n}{2}$$

And hence it don't contain any disc of radius 1. ■

COROLLARY. *$f$ is an entire function that does not fix any point then $f \circ f$ has a fixed point unless $f(z) = z + b$*

*Proof.* Consider the function, $g(z) = \frac{f \circ f(z) - z}{f(z) - z}$ misses 0 and 1, hence by Picard theorem we can say it is a constant function. Thus,

$$f(f(z)) - z = c(f(z) - 1)$$
$$\Rightarrow f'(f(z))f'(z) - 1 = c(f'(z) - 1)$$
$$\Rightarrow f'(z)(f'(f(z)) - c) = 1 - c$$

(complete the proof)

COROLLARY. *If $f$ and $g$ are entire and $f^n + g^n = 1$ then $f, g$ are either constant or they have common poles.*
*Proof.*

# Proof of Bloch's Theorem

## Lemma 1

Let, $G \subseteq \mathbb{C}$ bounded and $f : G \to \mathbb{C}$ continuous suppose $f|_G$ is open. Let, $a \in G$ be such that $s = \min_{s \in \partial G} |f(z) - f(a)|$ then $f(G) \subseteq B(f(a), s)$
*Proof.* It is an exercise from Topology.

## Lemma 2

Let, $V = B(a, r)$ and let $f \in \mathbf{Hol}(V)$ be non-constant. If $\|f'\|_V \leq 2|f'(a)|$, then

$$f(V) \supseteq B(f(a), R)$$

where $R = (3 - 2\sqrt{2})r|f(a)|$.

*Proof.* Consider the following function

$$A(z) = f(z) - f'(0)z = \int_{[0,z]} (f'(w) - f'(z))dw$$

Using Cauchy Integral formula we can say,

$$f'(v) - f'(0) = \frac{v}{2\pi i} \oint_{\partial V} \frac{f'(w)}{w(w-v)} dw$$

Now we have,

$$\begin{aligned} |A(z)| &\leq \int_0^1 |f'(zt) - f'(0)||z|\, dt \\ &\leq \int_0^1 \frac{|zt| \|f\|_V}{r - |zt|} |z|\, dt \\ &\leq \frac{|z|^2}{(r - |z|)} \|f'\|_V \end{aligned}$$

This $0 < \rho < r$, for $|z| = \rho$, $|f(z)| \geq \left(\rho - \frac{\rho^2}{r - \rho}\right)|f'(0)|$

## Main Proof

Let, $f \in \mathbf{Hol}(E)$ it means $f(E) \supseteq B(f(a), (3/2 - 2\sqrt{2})M)$. Note, $|f'(z)|(1 - |z|)$ is continuous on $\bar{E}$. Suppose $p \in E$ be the maxima point of the above function. And let $M$ be the maximum then $(3/2 - \sqrt{2})M > 1/12|f'(0)|$. Let, $t = \frac{1 - |p|}{2}$, this means $M = 2t|f'(p)|$ also $B(p, t) \subseteq E$ and $1 - |z| \geq t$ (via triangle inequality)