Complex Analysis

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Contents

Lecture 1

1.1 Introduction

Our objective is to study functions $f : \mathbb{C} \to \mathbb{C}$. We know that as metric spaces $\mathbb{C}$ and $\mathbb{R}^2$ are isometric, with the natural map $(x, y) \mapsto x + iy$ being an isometry, but then what is the difference between analysis in $\mathbb{R}^2$ and analysis in $\mathbb{C}$? The difference arises because $\mathbb{C}$ is a field while $\mathbb{R}^2$ is not a field, thus we have a notion of multiplication and division in the complex plane.
Before going into further details we recall some of the obvious observations that one can make,

1. (Triangle inequality). $||z_1| - |z_2|| \leq |z_1 - z_2|$.

2. $|z| \geq \max\{|x|, |y|\}$ where $z = x + iy$.

3. $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is a Cauchy sequence if and only if $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^2$ is a Cauchy sequence, which is equivalent to $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are Cauchy sequences.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Then f is continuous (or limit exists) at a point $z_0 = x_0 + iy_0$ if and only if $f : \mathbb{R}^2 \to \mathbb{R}^2$ viewed as a function from the real plane to the real plane is continuous (or limit exists) at (x_0, y_0).

We haven't yet clarified how the analysis of $\mathbb{C}$ differs from the analysis of $\mathbb{R}^2$, the fact that $\mathbb{C}$ is a field gives us that $\frac{f(z)-f(z_0)}{z-z_0} \in \mathbb{C}$ for all $z \neq z_0$. Thus we can define the derivative of $f : \mathbb{C} \to C$ at z_0 as the complex number obtained by taking the limit $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ (provided the limit exists). Thus the derivative of $f : \mathbb{C} \to C$ at $z_0 \in \mathbb{C}$ is a complex number, while the derivative of $f : \mathbb{R}^2 \to \mathbb{R}^2$ (that is, the total derivative) is a 2×2 matrix.
This raises the following question let $f = u + iv$, then if we view $f = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ we know

$$J_f(x_0, y_0) = Df(x_0, y_0) = \begin{bmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{bmatrix},$$

is there any relation between $f'(z_0)$ (if it exists) and $J_f(x_0, y_0)$?

Homewok. (another representation of $\mathbb{C}$). Let

$$M = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \mid x, y \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R}) \tag{1.1}$$

Show that M is a field under matrix multiplication and is in fact isomorphic to $\mathbb{C}$.

The above assignment suggests there must be some representation of $f'(z_0)$ in terms of the Jacobian matrix $J_f(x_0, y_0)$, indeed there is some relation which we will discuss in a while.
Notation. $B_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$.

Definition 1.1.1 ▶ Holomorphic Functions.

Let $\mathcal{O}$ be an open subset of $\mathbb{C}$, and let $f : \mathcal{O} \to \mathbb{C}$ be a function and $z_0 \in \mathcal{O}$. We say that f is **$\mathbb{C}$-differentiable at** z_0 or **holomorphic at** z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \text{ exists.}$$

And we will say f is **holomorphic on** $\mathcal{O}$ if f is holomorphic at every point $z \in \mathcal{O}$. We will denote by

$$\mathbf{Hol}(\mathcal{O}) = \{f : \mathcal{O} \to \mathbb{C} \mid f \text{ is holomorphic}\}.$$

Then $\mathbf{Hol}(\mathcal{O})$ forms an algebra over $\mathbb{C}$.

Lemma 1.1.1 (Some Immediate Observations.)

Let $f, g : \mathcal{O} \to \mathbb{C}$ be holomorphic at z_0, then

1. f is continuous at z_0.
2. $(\alpha f + g)'(z_0) = \alpha f'(z_0) + g'(z_0)$ for all $\alpha \in \mathbb{C}$.

Example 1.1.1

Some examples of holomorphic functions are $f(z) = z$, $f(z) =$ constant and $f(z) = z^2$, while $f(z) = \bar{z}$ is not a holomorphic function. Note that in $\mathbb{R}^2$ the function $f(z) = \bar{z}$ corresponds to the function $f(u, v) = (u, -v)$. But then we get that $Df(u, v) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \notin M$ (where M is defined in equation 1.1). If we now consider the function $f(z) = z^2$, then in $\mathbb{R}^2$ it corresponds to the function $f(x, y) = (x^2 - y^2, 2xy)$ then $J_f(x, y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix} \in M$.

The above example gives us the motivation to answer the problem we had raised earlier: *how are the complex derivative and the Jacobian matrix related?*

1.2 Holomorphic versus Differentiable Functions.

For this discussion we will let $f = u + iv : \mathcal{O} \to \mathbb{C}$, let $z_0 = x_0 + iy_0$. Suppose f is holomorphic at z_0, and let $\alpha = a + ib = f'(z_0)$. We then define the function for all $z \in B_r(z_0)$

$$\begin{aligned} R(z) &= f(z) - f(z_0) - \alpha(z - z_0) \\ &= \underbrace{[u(z) - u(z_0) - a(x - x_0) + b(y - y_0)]}_{R_1(z)} + i \underbrace{[v(z) - v(z_0) - b(x - x_0) - a(y - y_0)]}_{R_2(z)}. \end{aligned}$$

Now recall that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at (x_0, y_0) if and only if

$$\frac{R(x, y)}{\|(x, y) - (x_0, y_0)\|} \to 0 \text{ as } (x, y) \to (x_0, y_0).$$

But we have

$$\frac{R(z)}{|z - z_0|} = \frac{R_1(z)}{|z - z_0|} + i \frac{R(z)}{|z - z_0|},$$

and we also know that f is holomorphic at z_0 hence we get that

$$\lim_{z \to z_0} \frac{R(z)}{|z - z_0|} = 0 \iff \lim_{z \to z_0} \frac{R_1(z)}{|z - z_0|} = \lim_{z \to z_0} \frac{R_2(z)}{|z - z_0|} = 0.$$

Thus it is equivalent to saying that $u, v : \mathcal{O} \to \mathbb{R}$ are differentiable at (x_0, y_0) and we further have

$$\begin{aligned} a &= u_x = v_y \\ b &= v_x = -u_y. \end{aligned}$$

> **Theorem 1.2.1 (Cauchy Riemann Equations)** Let $f := u + iv : \mathcal{O} \to \mathbb{C}$ be a function and $z_0 \in \mathcal{O}$. Then f is holomorphic at z_0 if and only if $u, v : \mathcal{O} \to \mathbb{R}$ is differentiable at z_0 and $u_x = v_y$ and $u_y = -v_x$. These are called the Cauchy Riemann Equations, thus we have
>
> $$u_x = v_y,\ u_y = -v_x \quad \text{and} \quad f'(z_0) = u_x(z_0) + iv_x(z_0).$$

Lecture 2

2.1 Complex partial differential operators

Definition 2.1.1 ▶ Complex C^k functions

Let $\mathcal{O} \subset \mathbb{R}^2$ be open. We say $f : \mathcal{O} \to \mathbb{C}$ is in $C^1(\mathcal{O})$ if the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous on $\mathcal{O}$. Similarly, we say $f \in C^k(\mathcal{O})$ if all the partial derivatives of order $\leq k$ are continuous on $\mathcal{O}$, that is,

$$C^k(\mathcal{O}) = \left\{ f : \mathcal{O} \to \mathbb{C} \;\middle|\; \frac{\partial^t f}{\partial x^i \partial y^j} \text{ is continuous on } \mathcal{O} \text{ for all } i + j = t \text{ and } 1 \leq t \leq k \right\}.$$

We denote continuous functions as $C^0(\mathcal{O})$ or $C(\mathcal{O})$. Note that the notion of continuity is independent of whether f is treated as a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ or $f : \mathbb{C} \to \mathbb{C}$.

Definition 2.1.2 ▶ Complex partial derivatives

Let $f = u + iv : \mathcal{O} \to \mathbb{C}$ be in $C^1(\mathcal{O})$. Then we define the **complex partial derivatives** of f as,

$$\begin{aligned} \partial f = \frac{\partial f}{\partial z} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv) \\ \text{and } \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} &:= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) \end{aligned}$$

Note that ∂f and $\bar{\partial} f$ exist when the partials $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist. We don't need the continuity of the partials to define the complex partial derivatives. In practice however, we will almost always end up working with C^1 functions. Consider the following example,

Example 2.1.1

Let $f(z) = z$. Then,

$$\begin{aligned} \partial f = \frac{\partial z}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x + iy) \\ &= \frac{1}{2}(1 - i^2) = 1 \end{aligned}$$

Also for $g(z) = \bar{z}$,

$$\partial g = \frac{\partial \bar{z}}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x - iy)$$

$$= \frac{1}{2}(1 + i^2) = 0$$

We collect some properties of these differential operators in the following lemma. The proofs follows from definition.

Lemma 2.1.1

Let $f, g \in C^1(\mathcal{O})$ and $\alpha, \beta \in \mathbb{C}$ be scalars. Then show that,

1. $\partial(\alpha f + \beta g) = \alpha \partial f + \beta \partial g$.
2. $\bar{\partial}(\alpha f + \beta g) = \alpha \bar{\partial} f + \beta \bar{\partial} g$.
3. $\partial(fg) = f \partial g + g \partial f$.
4. $\bar{\partial}(fg) = f \bar{\partial} g + g \bar{\partial} f$.

Remark. For $f = u + iv \in C^1(\mathcal{O})$, we have,

$$\begin{aligned} \bar{\partial f} = \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) \\ &= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y) \end{aligned}$$

so it is immediate that, $f \in \mathbf{Hol}(\mathcal{O})$ if and only if $\bar{\partial f} = 0$ on $\mathcal{O}$. Again, for $f \in \mathbf{Hol}(\mathcal{O})$, $\partial f = f'$ on $\mathcal{O}$. We will see that these differential operators will have roles analogous to the operators ∂_x and ∂_y over $\mathbb{R}^2$.

Theorem 2.1.1 Let $\mathcal{O} \subseteq \mathbb{C}$ be a open and connected, and $f = u + iv \in \mathbf{Hol}(\mathcal{O})$.

1. If $f' = 0$ then f is constant on $\mathcal{O}$.
2. If $f(\mathcal{O}) \subseteq \mathbb{R}$ then f is constant on $\mathcal{O}$.

Proof. 1. Let $f' = 0$ on $\mathcal{O}$. Then $u_x = 0$ and $v_x = 0$. So u and v are $x-$free. Also, by the Cauchy-Riemann equations, $u_y = v_x = 0$ and $v_y = -u_x = 0$. So u and v are $y-$free. So, by connectedness of $\mathcal{O}$, u and v are constant. Hence, f is constant on $\mathcal{O}$.

2. Let $f(\mathcal{O}) \subseteq \mathbb{R}$. Then v is constant. So $v_x = v_y = 0$. And by the Cauchy-Riemann equations, $u_x = v_y = 0$ and $u_y = -v_x = 0$. Sou is constant. Hence, f is constant on $\mathcal{O}$.

□

This indicates that the notion of holomorphicity is quite fundamentally different from that of the $\mathbb{R}^2$-derivative, and any non-trivial examples requires the functions to be complex valued.

2.2 Harmonic functions

This is a very interesting class of functions, frequently encountered in the theory of PDEs and in complex analysis. These are precisely the solution to the PDE $\Delta f = 0$, where Δ is the appropriate **Laplacian operator**. For functions over $\mathbb{R}^2$, this equation becomes

$$\Delta f = f_{xx} + f_{yy} = 0$$

Let, $f = u + iv \in \mathbf{Hol}(\mathcal{O}) \cap C^2(\mathcal{O})$. Then the Cauchy-Riemann equations are

$$u_x = v_y \text{ and } u_y = -v_x$$

Now, we take the partial derivative of the first equation with respect to x, and that of the second one with respect to y. As the functions $u, v \in C^2(\mathcal{O})$, the mixed partial derivatives are independent of the order of integration. Thus,

$$u_{xx} = v_{yx} = v_{xy} \text{ and } u_{yy} = -v_{xy} = -v_{yx}$$

Adding these two equations gives $u_{xx} + u_{yy} = 0$, that is, u is harmonic. Similarly, $v_{xx} + v_{yy} = 0$ and therefore v is also harmonic. Henceforth, the Laplacian Operator will be defined as

$$\Delta \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Definition 2.2.1 ▶ Harmonic function

A function $f \in C^2(\mathcal{O})$, where $\mathcal{O} \subseteq \mathbb{R}$, is called **harmonic** if $\Delta f = 0$.

Thus the real and imaginary parts of any holomorphic function are harmonic. This raises the obvious question as to whether the converse is true. *Is every harmonic function the real part of some holomorphic function?* The answer depends on the domain of definition of the functions. In sufficiently nice domains, this is indeed the case. However, there are subsets of the complex plane where this cannot be done. Answering this question, particularly for slightly more general situation $\Delta f = Ef$, takes us to the theory of partial differential equations. We will develop some machinery and then hopefully come back to this question.

2.3 Integration of complex functions

We already have a notion of line integrals over $\mathbb{R}^2$. Naturally, the question arises, whether integration in $\mathbb{C}$ can be defined in an analogous manner. That is the journey we embark upon now.

Line integrals

Definition 2.3.1 ▶ Parametrized curve

A **parametrized curve** is a continuous function, often denoted by $\gamma, z : [a, b] \to \mathbb{C}$. We will separate the real and imaginary parts as $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ or as or $z(t) = x(t) + iy(t)$.

- We say γ is **closed** if $\gamma(a) = \gamma(b)$.
- γ is said to be **simple closed** if γ is closed and one-one on $[a, b)$.

Definition 2.3.2

A function $\gamma : [a, b] \to \mathbb{C}$ is said to be in $C^1[a, b]$ if both $\mathrm{Re}(\gamma)$ and $\mathrm{Im}(\gamma)$ are in $C^1[a, b]$.

We can talk about integration of a continuous over any curve, in a manner analogous to the definition of Riemann Integration. However, in most examples, integration will be carried out on C^1 curves, in which case, (2.1) holds. So strictly speaking, Definition - 2.3.4 is a consequence rather than a definition. More details regarding the same can be found in the next lecture.

Definition 2.3.3 ▶ Integration of a curve

Let $\gamma : [a, b] \to \mathbb{C}$ be a curve. Then we define the **integral of** γ as,

$$\int_a^b \gamma(t) \, \mathrm{d}t = \int_a^b \gamma_1(t) \, \mathrm{d}t + i \int_a^b \gamma_2(t) \, \mathrm{d}t$$

Definition 2.3.4 ▶ Contour integral

Let $\gamma : [a, b] \to \mathbb{C}$ be a C^1 curve and $f \in C(\{\gamma(t) : t \in [a, b]\})$ be a function. Then we define the **line integral** or **contour integral of f along** γ as,

$$\int_\gamma f = \int_\gamma f(z)\ \mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\ \mathrm{d}t \tag{2.1}$$

The integral in the right hand side is an ordinary line integral over $\mathbb{R}^2$. However, for this to be a valid definition, it must be defined irrespective of the parametrization γ. This issue is addressed in the next lecture. For now, we look at some instructive examples.

Example 2.3.1

We now compute the integral of $f(z) = z^2$ over an arc the circle $\partial B_r(0)$ of radius $r > 0$ centered at 0. An arc of this circle can be parametrized as $\gamma(t) = re^{it}$ for $0 \le a \le t \le b \le 2\pi$. Then,

$$\begin{aligned}\int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\ \mathrm{d}t \\ &= \int_a^b (re^{it})^2(ire^{it})\ \mathrm{d}t \\ &= r^3 \int_a^b e^{3it}\ \mathrm{d}t\end{aligned}$$

Splitting the integral $\int_a^b e^{3it}\ \mathrm{d}t$ into real and imaginary parts, we get,

$$\begin{aligned}\int_a^b e^{3it}\ \mathrm{d}t &= \int_a^b \cos(3t)\ \mathrm{d}t + i\int_a^b \sin(3t)\ \mathrm{d}t \\ &= \left.\frac{\sin(3t)}{3}\right|_a^b + i \left.-\frac{\cos(3t)}{3}\right|_a^b \\ &= \left.\frac{e^{3it}}{3}\right|_a^b\end{aligned}$$

$$\begin{aligned}\implies \int_\gamma f &= \frac{r^3}{3}\left(e^{3ib} - e^{3ia}\right) \\ &\begin{cases} = 0 & \text{if } b - a = \frac{2n\pi}{3} \text{ for any } n \in \mathbb{Z} \\ \neq 0 & \text{otherwise}\end{cases}\end{aligned}$$

Consider a polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$. It is an immediate observation that the integral $\int_{\partial B_r(0)} p\, dz$ is always zero for any $r > 0$. This somehow indicates to the fact that $\int_{\partial B_r(0)} f\, dz = 0$ for some other "good" functions f, which exhibit a polynomial like behavior. The obvious guesses are functions which a power series expansion around a neighbourhood, i.e., analytic functions. This and much more will turn out to be true, but for now, we look at some other examples.

Example 2.3.2

Let γ be the line joining 1 and $1+2i$, which can be parametrized as $\gamma(t) = 1+2it$ for $0 \leq t \leq 1$. Then, we compute the integral of $f(z) = z^2$ over γ as,

$$\begin{aligned}\int_\gamma f &= \int_0^1 (1+2it)^2(2i)\ \mathrm{d}t \\ &= \frac{2}{3} + 4i \neq 0\end{aligned}$$

Example 2.3.3

Let $r > 0$ and $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be the fucntion $f(z) = \frac{1}{z}$. Note that, f is continuous on $\partial B_r(0)$ but $f \notin \mathbf{Hol}(B_r(0))$. We compute the integral of f over the circle $\partial B_r(0)$ with our previous parameterization as,

$$\begin{aligned}\int_{\partial B_r(0)} \frac{1}{z}\ \mathrm{d}z &= \int_0^{2\pi} \frac{1}{re^{it}}(ire^{it})\ \mathrm{d}t \\ &= \int_0^{2\pi} i\ \mathrm{d}t \\ &= 2\pi i \neq 0\end{aligned}$$

Writing differently,

$$\frac{1}{2\pi i}\int_{\partial B_r(0)} \frac{1}{z}\ \mathrm{d}z = 1$$

If we now increase the speed of parameterization of $\partial B_r(0)$ to n, i.e. $\gamma(t) = re^{int}$ for $0 \leq t \leq 2\pi$. Then we get

$$\frac{1}{2\pi i}\int_{\partial B_r(0)} \frac{1}{z}\ \mathrm{d}z = n$$

The integer n in the above example is called the **winding number** of the curve γ around 0, which is a topological invariant and is the starting point of index theory.

Lecture 3

3.1 More on contour integration

We briefly discuss the more fundamental development of integrals of complex functions over curves in the complex plane. Let $\gamma : [a, b] \to \mathcal{O}$ be a smooth curve in $\mathbb{C}$. Although rectifiability of the curve is a sufficient assumption for the development of the theory, we assume smoothness for the sake of brevity.
Let $P : a = t_0 < t_1 \cdots < t_n = b$ be a partition of $[a, b]$. Consider the points $z_j = \gamma(t_j), 0 \leq j \leq n$ and let $\gamma_j = \gamma([t_{j-1}, t_j])$ for $1 \leq j \leq n$. We pick tags $\zeta_j \in \gamma_j$ to get the tag set $T_P = \{\zeta_j\}_1^n$. Then, we define the mesh of the partition of P with respect to the curve γ to be

$$\|P\|_\gamma = \max\{|z_j - z_{j-1}| : 1 \leq j \leq n\}$$

The Riemann sum of the function f with respect to the partition P and the curve γ is defined to be

$$R(f, \gamma; P, T_P) = \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1})$$

We say that the function f is integrable over γ if the limit of R exists as $\|P\|_\gamma \to 0$. In that case, we define

$$\int_\gamma f \, \mathrm{d}z = \lim_{\|P\|_\gamma \to 0} R(f, \gamma; P, T_P)$$

Finally, we have the following result which justifies the preliminary discussion of the last lecture.

> **Theorem 3.1.1** If $f \in C(\mathcal{O})$, f is integrable over any smooth curve γ contained in $\mathcal{O}$ and
>
> $$\int_\gamma f \, \mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}t$$

We now discuss how contour integration of functions in $\mathbb{C}$ relate to integration of scalar fields in $\mathbb{R}^2$.
Let $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ be a smooth curve in $\mathbb{C}$ and $f = u + iv$. Then,

$$\begin{aligned}
f(\gamma(t))\gamma'(t) &= (u(\gamma(t)) + iv(\gamma(t)))(\gamma_1'(t) + i\gamma_2'(t)) \\
&= ((u \circ \gamma)(t)\gamma_1'(t) - (v \circ \gamma)(t)\gamma_2'(t)) + i((v \circ \gamma)(t)\gamma_1'(t) + (u \circ \gamma)(t)\gamma_2'(t)) \\
\implies \int_\gamma f \, \mathrm{d}z &= \int_a^b ((u \circ \gamma)(t)\gamma_1'(t) - (v \circ \gamma)(t)\gamma_2'(t)) \, \mathrm{d}t + i \int_a^b ((v \circ \gamma)(t)\gamma_1'(t) + (u \circ \gamma)(t)\gamma_2'(t)) \, \mathrm{d}t \\
\implies \int_\gamma f \, \mathrm{d}z &= \left(\int_\gamma u \, \mathrm{d}x - v \, \mathrm{d}y\right) + i\left(\int_\gamma v \, \mathrm{d}x + u \, \mathrm{d}y\right)
\end{aligned}$$

We can slightly abuse the notation to write the identity above suggestively as:

$$\int_\gamma f\,\mathrm{d}z = \int_\gamma f\,\mathrm{d}x + i\int_\gamma f\,\mathrm{d}y = \int_\gamma (f\,\mathrm{d}x + if\,\mathrm{d}y)$$

Exercise: Show that for any smooth curve γ, the operation of integrating functions over γ is linear, i.e, $\int_\gamma$ is a linear functional on the space of all functions that are integrable over γ.

Definition 3.1.1

For $\gamma : [a,b] \to \mathbb{C}$ a smooth curve, we define the length of γ to be

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\,\mathrm{d}t$$

Definition 3.1.2

A curve $\gamma : [a,b] \to \mathbb{C}$ is said to be piecewise smooth if there is a partition $P : a = t_0 < t_1 \cdots t_n = b$ of $[a,b]$ such that $\gamma\big|_{[t_{j-1},t_j]}$ is smooth, for all $1 \leq j \leq n$.

The preceding constructions are easily generalised to piecewise smooth curves, simply by breaking up the curve into its smooth parts and integrating separately over each such part and adding the results. We leave it as an easy exercise for the reader to formulate the exact expressions.

Exercise: If $-\gamma$ denotes the curve $t \mapsto \gamma(a+b-t)$ for $\gamma : [a,b] \to \mathbb{C}$ a (piecewise) smooth curve, then

$$\int_{-\gamma} f\,\mathrm{d}z = -\int_\gamma f\,\mathrm{d}z$$

Theorem 3.1.2 Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve.

(1) $\left|\int_a^b \gamma(t)\,\mathrm{d}t\right| \leq \int_a^b |\gamma(t)|\,\mathrm{d}t$

(2) for all f that is continuous on the range of γ,

$$\left|\int_\gamma f\,\mathrm{d}z\right| \leq \sup_{t\in[a,b]} |f(\gamma(t))|\ell(\gamma)$$

Proof. (1) Let $\alpha = \int_a^b \gamma(t)\,\mathrm{d}t$, and assume without loss of generality that $\alpha \neq 0$. Consider $\beta = \frac{\alpha}{|\alpha|}$,

$$|\alpha| = \beta\int_a^b \gamma(t)\,\mathrm{d}t = \mathrm{Re}\left\{\int_a^b \beta\gamma(t)\,\mathrm{d}t\right\} \leq \int_a^b |\mathrm{Re}\{\beta\gamma(t)\}|\,\mathrm{d}t$$

$$\implies \left|\int_a^b \gamma(t)\,\mathrm{d}t\right| \leq \int_a^b |\beta\gamma(t)|\,\mathrm{d}t = \int_a^b |\gamma(t)|\,\mathrm{d}t$$

Note that we only require γ to be a (continuous) curve for this part.

(2) We have,

$$\left|\int_\gamma f\,\mathrm{d}z\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t\right| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|\,\mathrm{d}t \qquad \text{(by the first part)}$$

$$\implies \left|\int_\gamma f\,\mathrm{d}z\right| \leq \sup_{t\in[a,b]} |f(\gamma(t))|\ell(\gamma)$$

as was to be shown.

□

The following result is another chain rule, applicable for functions evaluated over curves.

Theorem 3.1.3 Let $f \in \mathsf{Hol}(\mathcal{O}), \gamma : [a, b] \to \mathcal{O}$ a C^1 curve. Then,

$$(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t)$$

for all $t \in (a, b)$.

Proof. Let $f = u + iv$ and $\gamma(t) = x(t) + iy(t)$. Then,

$$\begin{aligned}
(f \circ \gamma)'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(u(x(t), y(t)) + iv(x(t), y(t))) \\
&= \left(u_x \frac{\mathrm{d}x}{\mathrm{d}t} + u_y \frac{\mathrm{d}y}{\mathrm{d}t}\right) + i\left(v_x \frac{\mathrm{d}x}{\mathrm{d}t} + v_y \frac{\mathrm{d}y}{\mathrm{d}t}\right) \\
&= \left(u_x \frac{\mathrm{d}x}{\mathrm{d}t} - v_x \frac{\mathrm{d}y}{\mathrm{d}t}\right) + i\left(v_x \frac{\mathrm{d}x}{\mathrm{d}t} + u_x \frac{\mathrm{d}y}{\mathrm{d}t}\right) && \text{(Cauchy-Riemann equations)} \\
&= (u_x + iv_x)\left(\frac{\mathrm{d}x}{\mathrm{d}t} + i\frac{\mathrm{d}y}{\mathrm{d}t}\right) \\
\implies (f \circ \gamma)'(t) &= f'(\gamma(t))\gamma'(t)
\end{aligned}$$

□

Definition 3.1.3

We say that $f \in \mathsf{Hol}(\mathcal{O})$ has a primitive if there is $g \in \mathsf{Hol}(\mathcal{O})$ such that $g' = f$.

Theorem 3.1.4 Let $f \in \mathsf{Hol}(\mathcal{O})$ and $\gamma : [a, b] \to \mathcal{O}$ be a smooth curve. Assume f' is continuous in $\mathcal{O}$. Then,

$$\int_\gamma f' \,\mathrm{d}z = f(\gamma(b)) - f(\gamma(a))$$

Proof. We have,

$$\int_\gamma f' \,\mathrm{d}z = \int_a^b f'(\gamma(t))\gamma'(t) \,\mathrm{d}t = \int_a^b (f \circ \gamma)'(t) \,\mathrm{d}t$$

We now split the derivative $(f \circ \gamma)'$ into its real and imaginary parts, and use the fundamental theorem of calculus to conclude what was required. □

Corollary

Let $\gamma : [a, b] \to \mathbb{C}$ be a closed smooth curve and $f \in \mathsf{Hol}(\mathcal{O})$, where $\gamma([a, b]) \subseteq \mathcal{O}$. Then,

$$\int_\gamma f' \,\mathrm{d}z = 0$$

This corollary can be used to show that certain functions cannot admit any primitives in a given domain, as in the following example.

Example 3.1.1

Recall that

$$\frac{1}{2\pi i} \int_{\partial B_r(0)} \frac{1}{z} \,\mathrm{d}z = 1$$

for all $r > 0$. Hence, even though $z \mapsto \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$, it does not admit any primitive in this domain!

Lecture 4

4.1 Cauchy Integral Theorem

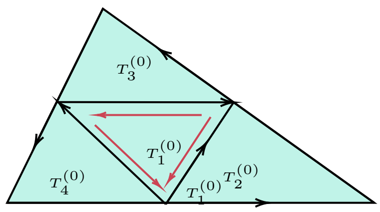
Last day we have seen, if a function have primitive over a simply connected domain, its' integral over the boundary of that region is 0. We also noted for a function $\frac{1}{z}$, integration (countour integral) over $\partial B_r(0)$ is not zero. So, it can't have a primitive over the region $\mathbb{C} \setminus \{0\}$. The function is Holomorphic on any open set do not contain 0. So the natural question arises,'Is it possible that, this function has primitive in some open set do not contain 0?' To answer this question we need 'Cauchy Integral Theorem', which is stated as following.

> **Theorem 4.1.1 (Cauchy(Goursat) Integral Theorem)** Let, $f \in \mathbf{Hol}(\mathcal{O})$, where $\mathcal{O}$ is simply connected domain. Then, $\int_\gamma f dz = 0$ for all piece-wise smooth closed curve $\gamma \subset \mathcal{O}$.

To prove this we need to develop more techniques in complex analysis. Rather, we will prove a weak version of the above theorem.

> **Theorem 4.1.2 (Cauchy Integral Theorem Δ version)** Let, γ is a triangle (curve including sides of the triangle T) and $\operatorname{int}(T) \cup T \subseteq \mathcal{O}$. Then for all $f \in \mathbf{Hol}(\mathcal{O})$,
>
> $$\int_T f dx = 0$$

Proof. Lat, T^0 b the curve T with anticlockwise direction. Take the middle point of each side and joint them to get 4 triangles (as shown in the picture) with an orientation (shown in the picture). Call these triangles $T_j^{(0)}$ for $j = 1, \cdots, 4$.



Let, $I = \int_{T^{(0)}} f dz$. From the above partition we can say,

$$I = \int_{T^{(0)}} f dz = \sum_{j=1}^{4} \int_{T_j^{(0)}} f dz$$

$$|I| \leq \sum_{j=1}^{4} \left| \int_{T_j^{(0)}} f dz \right| \leq 4 \left| \int_{T_i^{(0)}} f dz \right|$$

The last inequality holds for some $i \in \{1, 2, 3, 4\}$. Now call this triangle $T_i^{(0)} := T^(1)$. We carry out the calculations for $i = 1, \cdots, n$ and obtain $\frac{1}{4^n}|I| \leq \left|\int_{T^{(n)}} f dz\right|$. Set, $\mathcal{T}^{(n)} := T^{(n)} \cup \operatorname{int}(T^{(n)})$. These sets are compact for $n \in \mathbb{N}$. Thus, we have a chain of compact (closed subsets),

$$\mathcal{T}^{(0)} \supset \mathcal{T}^{(1)} \supset \cdots$$

with $\operatorname{diam}\mathcal{T}^{(n)} = \frac{1}{2^n}\operatorname{diam}\mathcal{T}^{(0)} = \frac{1}{2^n}($ length of largest side of $\mathcal{T}^{(0)})$. Using "**Cantor intersection theorem**" we get,

$$\bigcap_{n=0}^{\infty}\mathcal{T}^{(n)} = \{z_0\}\,\text{(singleton set)}$$

Using holomorphic property of f at z_0 we get, $f(z) = f(z_0) + f'(z_0)(z-z_0) + R(z)(z-z_0)$ holds in some open ball $B_\epsilon(z_0)$ around z_0, where $R(z)$ is continuous on that open ball with $\lim R(z) = 0$ as $z \to z_0$. Take n large enough so that $\mathcal{T}^{(n)} \subset B_\epsilon(z_0)$.

$$\begin{aligned}\int_{T^{(n)}} f dz &= \int_{T^{(n)}} [f(z_0) + f'(z_0)(z-z_0) + R(z)(z-z_0)]dz \\ &= \int_{T^{(n)}} R(z)(z-z_0)dz \\ \implies \frac{1}{4^n}|I| &\leq \left|\int_{T^{(n)}} R(z)(z-z_0)dz\right|\end{aligned}$$

Set, $\epsilon_n := \sup_{z\in T^{(n)}}|R(z)|$. Note that as $n\to\infty$, $\epsilon_n \to 0$. Also, for $z \in \mathcal{T}^{(n)}$ we have, $|z-z_0| \leq \operatorname{diam}\mathcal{T}^{(n)} = \frac{1}{2^n}d_0$ (here, d_0 is the length largest side of $T^{(n)}$). Using triangle inequality we get,

$$\begin{aligned}\frac{1}{4^n}|I| \leq \left|\int_{T^{(n)}} R(z)(z-z_0)\,dz\right| &\leq \sup_{z\in T^{(n)}}|R(z)| \times \frac{d_0}{2^n} \times \operatorname{len}(T^{(n)}) \\ &= \frac{3d_0}{4^n}\cdot\epsilon_n \\ |I| &\leq 3d\epsilon_n\end{aligned}$$

Just by taking the limit $n\to\infty$ we have $I = 0$. ■

Corollary. *If R is a rectangle $R \subset \mathcal{O}$ and $f \in \mathbf{Hol}(\mathcal{O})$, then $\int_R f dz = 0$.*

Now we will look at the question, we concerned at the beginning. 'Does the primitive of $\frac{1}{z}$ exist in some domain $0 \notin B_\epsilon(z_0)$?' The answer is **Yes**! It should be given by $g(z) := \int_{\gamma_z} f\,dz$ where γ_z is a path from z_0 to z. The natural question should be to prove the well define-ness of the above g. We have to show, no matter what path b/w z and z_0 we choose, value of $g(z)$ must be same. If we have proven **main Cauchy theorem** then for any two path γ,η (that do not intersect each other except the end points) joining z and z_0, we will consider the concatenated loop $\gamma * \eta^{-1}$. Since the disc is simply-connected, the region bounded by the loop is also simply connected.Thus the theorem gives, $\int_{\gamma*\eta^{-1}} f\,dz = 0$ which gives $\int_\gamma f = -\int_{\eta^{-1}} f\,dz = \int_\eta f\,dz$. Thus, the function we defined is well-defined. Since we haven't proved the strong Cauchy theorem, we will answer the question with the help of following theorem.

> **Theorem 4.1.3** Let, $\mathcal{U} \subseteq \mathcal{O}$ open and convex. $f \in \operatorname{Cont}(\mathcal{U})$ and suppose $\int_{\partial\Delta} f\,dz = 0$ for all solid triangles $\Delta \subseteq \mathcal{U}$. Fix $z_0 \in \mathcal{U}$ and let $[z_0, z]$ be the line joining z_0 to z. Define,
>
> $$g(z) := \int_{[z_0,z]} f\,dz$$
>
> Then $g \in \mathbf{Hol}(\mathcal{U})$ and $g' = f$.

Proof. Fix $\tilde{z} \in \mathcal{U}$ and T_z be the triangle with sides $[z_0,\tilde{z}], [\tilde{z}, z], [z, z_0]$ (with this orientation). By the property of f we can say, $\int_{T_z} f\,dz = 0$. Expanding this integral we get,

$$\begin{aligned}&\Rightarrow \int_{[z_0,\tilde{z}]} f\,dz + \int_{[\tilde{z},z]} f\,dz + \int_{[z,z_0]} f\,dz = 0 \\ &\Rightarrow g(\tilde{z}) - g(z) = \int_{[z,\tilde{z}]} f\,dz\end{aligned}$$

$$\Rightarrow \frac{g(z) - g(\tilde{z})}{z - \tilde{z}} = \frac{1}{z - \tilde{z}} \int_{[z,\tilde{z}]} f \, dz$$

$$\Rightarrow \left| \frac{g(z) - g(\tilde{z})}{z - \tilde{z}} - f(\tilde{z}) \right| = \frac{1}{|z - \tilde{z}|} \left| \int_{[z,\tilde{z}]} f \, dz \right|$$

Using continuity of f we can say RHS of the above equation is $< \epsilon$ in some open nbd around $\tilde{z}$. Thus, g is holomorphic at $\tilde{z}$ with $f(\tilde{z}) = g'(\tilde{z})$, we can do this for any $\tilde{z} \in \mathcal{U}$ and hence we are done. ■

Corollary. *Let, $f \in \mathbf{Hol}(\mathcal{U})$ (with $\mathcal{U}$ is open and convex). Then there exist $g \in \mathbf{Hol}(\mathcal{U})$ such that $g' = f$ and $\int_\gamma f \, dz = 0$ for all smooth piece-wise smooth loop γ in $\mathcal{U}$.*

"One big theorem at a day"- *JDS*

Lecture 5

5.1 Stokes' and Green's Theorem

Before developing some more tools required to prove the Cauchy-Goursat theorem, we give some more motivation towards the result; what happens for C^1 functions. Recall Stokes's theorem for $\mathbb{R}^2$: Consider a simply connected region $\Omega \subseteq \mathbb{R}^2$ with piecewise smooth, simple and closed boundary $\partial\Omega$. If $f = P\,\mathrm{d}x + Q\,\mathrm{d}y$ is a C^1 1−form on an open set $U \supseteq \Omega \cup \partial\Omega$, then

$$\oint_{\partial\Omega} f = \iint_{\Omega} \mathrm{d}f\,.$$

Now, as $f = P\,\mathrm{d}x + Q\,\mathrm{d}y$, we get $\mathrm{d}f = \mathrm{d}P\,\mathrm{d}x + \mathrm{d}Q\,\mathrm{d}y = P_y\,\mathrm{d}y\,\mathrm{d}x + Q_x\,\mathrm{d}x\,\mathrm{d}y = (Q_x - P_y)\,\mathrm{d}x\,\mathrm{d}y$. Hence,

$$\oint_{\partial\Omega} f = \oint_{\partial\Omega} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

which is nothing but the statement of Green's theorem. We now return to $\mathbb{C}$ and assume $f \in \mathsf{Hol}(\Omega) \cap C^1(\Omega)$, where Ω is a simply connected region as above. Let $\gamma \subseteq \Omega$ be a piecewise smooth, simple and closed curve. Then, by the discussion of lecture 3,

$$\oint_{\gamma} f\,\mathrm{d}z = \left(\oint_{\gamma} u\,\mathrm{d}x - v\,\mathrm{d}y\right) + i\left(\oint_{\gamma} v\,\mathrm{d}x + u\,\mathrm{d}y\right)$$

Using Stokes's theorem on the two integrals on the right we get, [Σ_γ is the capping surface of γ, in some lecture we have used the notion of capping surface as int (γ)]

$$\oint_{\gamma} f\,\mathrm{d}z = \iint_{\Sigma\gamma} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y + i\iint_{\Sigma\gamma} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = 0$$

where the final equality follows from the Cauchy-Riemann equations. This is exactly the statement of the Cauchy-Goursat theorem! We now set to develop the tools needed to remove the C^1 restriction, that is, to show that holomorphic functions must be C^1 on "nice" domains.
Exercise: Suppose $f \in C^1(\Omega)$, where Ω is as above. If γ is a curve as above, show that

$$\oint_{\gamma} f\,\mathrm{d}z = 2i \iint_{\Sigma\gamma} \overline{\partial} f\,\mathrm{d}x\,\mathrm{d}y$$

holds for all such functions, without assuming holomorphicity.

Recall the result from lecture 3 that

$$\frac{1}{2\pi i}\oint_{C_r(0)} \frac{1}{z} = 1$$

for any $r > 0$. Changing variables, we get the equality

$$\frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{1}{z - z_0} = 1$$

for any $z_0 \in \mathbb{C}$. We now generalise this result to the following very useful theorem.

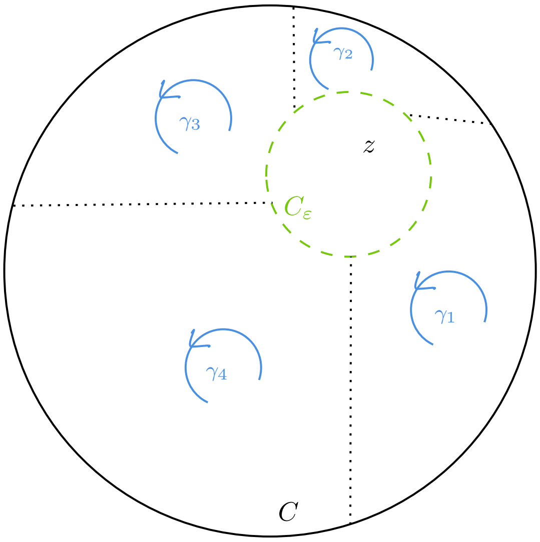
5.2 Cauchy Integral Formula

Theorem 5.2.1 (Cauchy Integral Formula) Let $f \in \mathsf{Hol}(\mathcal{O})$ and $C \subsetneq \mathcal{O}$ be a circle such that $D = C \cup \Sigma C$ is in $\mathcal{O}$. Then, for all $z \in \Sigma C$,

$$\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = f(z).$$

Remark. Note that if $z \in \mathcal{O} \setminus D$, by Theorem 4.1,

$$\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 0.$$



Proof. Fix $z \in \Sigma C$ and let $B_\varepsilon(z) \subsetneq \Sigma C$, $C_\varepsilon = \partial B_\varepsilon(z)$. We claim,

$$\oint_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \oint_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

Introduce 4 "cuts" as shown by the dotted lines in the figure, and let the loops displayed be oriented anticlockwise. Then, $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 = C \cup (-C_\varepsilon)$. Further,

$$\zeta \mapsto \frac{f(\zeta)}{\zeta - z}$$

is holomorphic in $\Sigma\gamma_j$ for each j. As each of these sets $\Sigma\gamma_j$ can be covered by an open convex set $\Omega_j \subsetneq \mathcal{O}$, we get by Theorem 4.1 that

$$\oint_{\gamma_j} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 0$$

for each j. Therefore, as integration over $\cup\gamma_j$ is simply the sum of each of these integrals, we get

$$\begin{aligned}
&\oint_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \oint_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\
\implies &\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - f(z) = \frac{1}{2\pi i}\oint_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \frac{f(z)}{2\pi i}\oint_{C_\varepsilon} \frac{1}{\zeta - z}\,\mathrm{d}\zeta \\
\implies &\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - f(z) = \frac{1}{2\pi i}\oint_{C_\varepsilon} \frac{f(\zeta) - f(z)}{\zeta - z}\,\mathrm{d}\zeta
\end{aligned}$$

By the triangle inequality,

$$\left|\frac{1}{2\pi i}\oint_{C_\varepsilon}\frac{f(\zeta)-f(z)}{\zeta-z}\,\mathrm{d}\zeta\right| \leq \frac{1}{2\pi}\sup_{\zeta\in C_\varepsilon}\left|\frac{f(\zeta)-f(z)}{\zeta-z}\right|\ell(C_\varepsilon) = \varepsilon\sup_{\zeta\in C_\varepsilon}\left|\frac{f(\zeta)-f(z)}{\zeta-z}\right|$$

As $\varepsilon \to 0$, the supremum goes to $|f'(z)|$ and so the right most quantity goes to 0. Therefore, in light of the equality above, we get

$$\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta - f(z) = 0$$

as was to be shown. □

Lecture 6

6.1 Some Remarks

In this lecture, we fisrt deal with some basic remarks regarding the Cauchy Integral Formula (Theorem 5.2.1). Then we will make some digression to the topic of power series in $\mathbb{C}$. We start by definng the notion of an entire function.

Definition 6.1.1

A function $f : \mathbb{C} \to \mathbb{C}$ is called entire if $f \in \mathbf{Hol}(\mathbb{C})$.

We give some simple illustrations to get a hold of Cauchy Integral Formula.

Example 6.1.1

To compute
$$\oint_{C_2(0)} \frac{ze^z}{z+i}\,\mathrm{d}z$$
we can use the Cauchy Integral Formula, to get
$$\oint_{C_2(0)} \frac{ze^z}{z+i}\,\mathrm{d}z = 2\pi i(-ie^{-i}) = 2\pi e^{-i}.$$

Example 6.1.2

To compute
$$\oint_{C_2(0)} \frac{ze^z}{z^2+1}\,\mathrm{d}z$$
we first note that $z^2 + 1 = (z + i)(z - i)$. Then we have
$$\begin{aligned}\oint_{C_2(0)} \frac{ze^z}{z^2+1}\,\mathrm{d}z &= \oint_{C_2(0)} \frac{ze^z}{(z+i)(z-i)}\,\mathrm{d}z \\ &= \frac{\pi}{2\pi i}\oint_{C_2(0)} \left(\frac{ze^z}{z-i} - \frac{ze^z}{z+i}\right)\mathrm{d}z \\ &= \pi\left(ie^i - (-ie^{-i})\right) \\ &= 2\pi i\cos(1).\end{aligned}$$

Now that we are in complex numbers, so we can factor polynomials. For example, we could factorise $z^2 + 1$ into $(z + i)(z - i)$. This was not possible with $\mathbb{R}$. So we can break any rational function into a product of linear factors and evaluate the integral using the Cauchy integral formula.

This immediately begs the question: *What are examples of holomorphic functions?* The answer is **Power series** ($\mathbb{C}[[X]]$). In fact these are all functions that are holomorphic.

Before we go into this, let us make a little digression for series in $\mathbb{C}$.

6.2 Series in $\mathbb{C}$

Given a sequence $\{\alpha_n\} \subseteq \mathbb{C}$, consider the "formal sum"

$$\sum_{j=0}^{\infty} \alpha_n \tag{6.1}$$

and define the *partial sums* by

$$S_n = \sum_{j=0}^{n-1} \alpha_n.$$

We say that (6.1) is summable if $\{S_n\}$ is a convergent sequence in $\mathbb{C}$. In this case, we define the sum of (6.1) to be $\lim_{n\to\infty} S_n$. Just like in $\mathbb{R}$, we can show that if (6.1) is summable (or converges), then $\{\alpha_n\}$ must converge to 0.

Example 6.2.1

For $z \in \mathbb{C}$, consider the geometric series

$$\sum_{n=0}^{\infty} \alpha^n$$

Then we have the partial sums

$$S_n = \sum_{k=0}^{n-1} \alpha^k = \frac{1-\alpha^n}{1-\alpha}.$$

which converges if and only if $|\alpha| < 1$. In this case, we have

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}. \tag{6.2}$$

Observe that the LHS of (6.2) is defined on $B_1(0)$ but the RHS is defined on $\mathbb{C} \setminus \{1\}$. This points to the notion of analytic continuation, which we will discuss in next lecture.
We now define the notion of absolute convergence in $\mathbb{C}$.

Definition 6.2.1 ▶ Absolute Convergence

The series $\sum_{n=0}^{\infty} \alpha_n$ is said to be absolutely convergent if $\sum_{n=0}^{\infty} |\alpha_n|$ is convergent.

This immediately raises the question: *What is the relationship between absolute convergence and convergence in $\mathbb{C}$?* The answer is infact similar to that in $\mathbb{R}$ and is given by the following theorem.

Theorem 6.2.1 (Comparison Test) Let $\{\alpha_n\}, \{\beta_n\} \subseteq \mathbb{C}$ be two sequencees such that $|\alpha_n| \leq |\beta_n|$ for all n. Then, if $\sum_{n=0}^{\infty} \beta_n$ converges absolutely then $\sum_{n=0}^{\infty} \alpha_n$ also converges absolutely.

Proof. Exercise. □

Definition 6.2.2 ▶ Power Series

Let $\{\alpha_n\} \subseteq \mathbb{C}$ be a sequence. Then the series

$$\sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$$

is called a power series in z about z_0 with coefficients $\{\alpha_n\}$.

Without loss of generality, we can assume that $z_0 = 0$ and consider the series

$$\sum_{n=0}^{\infty} \alpha_n z^n. \tag{6.3}$$

Remark.

- (6.3) always converges at $z = 0$.
- If (6.3) converges at $z = z' \neq 0$, then it converges absolutely for all z such that $|z| \leq |z'|$.
- If $\sum_{n=0}^{\infty} |\alpha_n||z|^n$ diverges at $z = z'$, then it diverges for all $|z| \geq |z'|$.

The above remarks raises a natural question: *What is the radius of convergence of* (6.3)*?* The answer is given by the following theorem.

Theorem 6.2.2 Given a power series $\sum_{n=0}^{\infty} \alpha_n z^n$, there exists a unique $R \in [0, \infty]$ (called the redius of convergence) such that

(i) The series converges absolutely on $B_R(0)$ ($= \mathbb{C}$ if $R = \infty$).

(ii) The series diverges for all z such that $|z| > R$.

Moreover, we have

(I) $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

(II) The series converges uniformly on all compact subsets of $B_R(0)$, eqiuvalently, For all small $\epsilon > 0$, the series converges uniformly on $\overline{B_{R-\epsilon}(0)}$.

Remark. If we have a unform convergent sequence of functions $\{f_n\}$ on a set $\mathcal{O}$. Then we say that $\int_\gamma f_n$ converges to $\int_\gamma f$ if $\operatorname{Im}\gamma \subset \mathcal{O}$. This would follow as $\int_\gamma$ can be treated as a line integral on $\mathbb{R}^2$, which in turn becomes a Riemann integral. Here lies one demonstration of the power of (II) of Theorem 6.2.2.

Proof of Theorem 6.2.2. We only prove (i) as similar proof will work for (ii). We set

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

and consider the following three cases:

- If $R = 0$, then $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = +\infty$. So there exists a subsequence of $\{a_n\}$ diverging to $+\infty$. Without loss of generality, we can assume that $\{a_n\}$ be that subsequence itself. Now fix $z \neq 0$, then there exists $N \in \mathbb{N}$ such that for all $n > N$, $\sqrt[n]{|a_n|} > \frac{1}{|z|}$, in other words, $|a_n||z^n| > 1$. This completes the proof.
- If $R = +\infty$, then $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0$. So for every $z \in \mathbb{C}$, $\limsup_{n\to\infty} |z| \sqrt[n]{|a_n|} = 0$. In other words, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|z| \sqrt[n]{|a_n|} < \frac{1}{2}$. This completes the proof by comparison test.

- For $0 < R < +\infty$, we fix $z \neq 0$ such that $|z| < R$. Then there exists $r > 0$ such that $|z| < r < R$, or eqiuvalently, $\frac{1}{|z|} > \frac{1}{r} > \frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. So there exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n| < \frac{1}{r^n}$, which yields $|a_n||z^n| < \frac{|z^n|}{r^n}$. As $\frac{|z|}{|r|} < 1$, comparison test shows that $\sum_{n=0}^{\infty} |a_n||z^n|$ converges. This completes the proof.

Finally, let $\epsilon > 0$ be arbitarily small such that $0 < R - \epsilon < R$, then $\sum_{n\geq N} |a_n|(R-\epsilon)^n \to 0$ as $N \to \infty$. So for every $z \in B_{R-\epsilon}(0)$,

$$\sum_{n\geq N} |a_n||z|^n \leq \sum_{n\geq N} |a_n|(R-\epsilon)^n$$

This shows the uniform convergence on $\overline{B_{R-\epsilon}(0)}$. □

Now we state a improtant lemma connecting the ratio and root tests and the convergence of power series.

Lemma 6.2.1

Let $\{a_n\} \subseteq \mathbb{C}$ be a sequence. Then we have the following,

$$\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \leq \liminf_{n\to\infty} \sqrt[n]{|a_n|} \leq \underbrace{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}_{\frac{1}{R}} \leq \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

Proof. Exercise. □

So whenever $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists we can easily compute the radius of convergence of the power series. In the next lecture, we will illustrate some simple examples of power series and their radius of convergence and then complete our discussion on the power series.

Lecture 7

In this lecture, we will complete our discussion of power series and consider analytic functions over the complex plane. We will then prove the equivalence of analytic and holomorphic functions - deriving the Cauchy integral formulae in the process. This is truly marvelous result with no immediate analogue in real analysis, and will be extremely useful in the discussions to follows.

7.1 Power Series continued

We started our discussion of power series over $\mathbb{C}$ in the previous lecture. We give some simple illustartions before moving on.

Example 7.1.1

Consider the power series given by $\sum_{n=1}^{\infty}(1+(-1)^n)z^n$. Then the radius of convergence for the same is given by

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{(1+(-1)^n)} = 1$$

Note that $\lim | \frac{a_{n+1}}{a_n} |$ does not exist here, so one cannot use the ratio test.

Example 7.1.2 (Exponential function)

Consider the power series $\sum_{n=1}^{\infty} \frac{z^n}{n!}$. Evidently, $\lim | \frac{a_{n+1}}{a_n} | = 0$, and thus the power series defines a function over $\mathbb{C}$. We call this the exponential function and denote it by $\exp\{z\}$. Also for $z_1, z_2 \in \mathbb{C}$, we have

$$\begin{aligned}
\exp\{z_1\}\exp\{z_2\} &= \left(\sum_{k=0}^{\infty}\frac{z_1^k}{k!}\right)\left(\sum_{l=0}^{\infty}\frac{z_2^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{z_1^r z_2^{n-r}}{r!(n-r)!} \\
&= \sum_{n=0}^{\infty}\frac{(z_1+z_2)^n}{n!} = \exp\{z_1+z_2\}
\end{aligned}$$

Note that the reordering of sums in justfied as the series is absolutely convergent. Using this multiplicative property, along with continuity of power series, one can show that

$$\exp\{z\} = e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!},$$

where $e := \exp\{1\} = \sum_{n=0}^{\infty} \frac{1}{n!}$. We will talk about this in more detail in the upcoming lectures.

Example 7.1.3 (Sine and Cosine functions)

We define the sine and cosine functions over $\mathbb{C}$ using the complex power series analogous to the Taylor series expansion of their real counterpart. In particular,

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \text{ and}$$
$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

Note that these power series absolutely converge over $\mathbb{C}$, and thus define entire functions. We also have the Euler's identity

$$e^z = \sum_{n=0}^{\infty} \frac{z_1^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$
$$= \cos(z) + i\sin(z)$$

We now comment on the differentiability of functions defined by power series. The exactly analogous result holds for real power series as well, and the proof is almost identical.

Theorem 7.1.1
Let R be the radius of convergence of the power series $f(z) := \sum a_n z^n$, where $z \in B_R(0)$. Then $f \in \mathbf{Hol}(\mathcal{O})$ and the derivative $f'(z) = \sum n a_n z^{n-1}$ is given by the correponding term-by-term differentiation, for $z \in B_R(0)$.

Proof. First we compute the radius of convergence R' of the derived power series $\sum n a_n z^{n-1}$. Evidently,

$$\frac{1}{R'} = \limsup_{n\to\infty} \sqrt[n]{n a_n} = \limsup_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{R}$$

Thus the derived series defines a function $g : B_R(0) \to \mathbb{C}$, such that

$$g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

Fix $z \in B_R(0)$ and correspondingly choose$\delta > 0$ such that $|z| + \delta < R$. Then for $|h| < \delta$,

$$\frac{f(z+h) - f(z)}{h} - g(z) = h \sum_{n=2}^{\infty} a_n p_n(z, h),$$

where $p_n(z,h) = \sum_{k=2}^{n} \binom{n}{k} h^{k-2} z^{n-k}$. The proof now follows as

$$\left| \frac{f(z+h) - f(z)}{h} - g(z) \right| \leq |h| \sum_{n=2}^{\infty} |a_n||p_n(z,h)| \leq |h| \sum_{n=2}^{\infty} |a_n| p_n(|z|, \delta)$$
$$\leq \frac{|h|}{\delta^2} \sum_{n=2}^{\infty} |a_n|(|z| + \delta)^n \longrightarrow 0 \text{ as } h \to 0$$

□

Thus, if R be the radius of convergence of a power series $f(z) := \sum a_n z^n$, where $z \in B_R(0)$, then $f \in C^\infty(B_R(0))$, where the power series coefficients are given by the corresponding term-by-term differentiation. Also, the coefficients $\{a_k\}$ can be computed from the derivatives of the function f at $z = 0$ as

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n z^{n-k}$$

$$\therefore a_k = \frac{f^{(k)}(0)}{k!}$$

7.2 Analytic Functions

Polynomials constitute the simplest examples of functions holomorphic on a given domain. The next, most prototypical examples are those which are locally generated by power series.

Definition 7.2.1 ▶ Analytic function

Let $f : \mathcal{O} \to \mathbb{C}$ be a continuous function, where $\mathcal{O}$ is open in $\mathbb{C}$. We say that f is analytic at $z_0 \in \mathcal{O}$ if there exists $r > 0$ such that f can be expressed as a convergent power series on $B_r(z_0)$. If this holds for all points $z_0 \in \mathcal{O}$, then we say f is analytic on $\mathcal{O}$.

From the results of the previous section, it is clear that analytic functions are holomorphic, and in fact, infinitely differentiable. For $f : \mathcal{O} \to \mathbb{C}$ analytic at z_0, the local power series representation is

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

The converse is decidedly false in the real case. Consider for instance, the function $f \in C^\infty(\mathbb{R})$, defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

It is easy to check that all derivatives of f vanish at $x = 0$, but f is not identically zero in any neighbourhood of $x = 0$. Thus, f is not analytic at $x = 0$. However this result has a positive answer for holomorphic functions; it truly is one of the most remarkable results of complex analysis.

Theorem 7.2.1 Let $\mathcal{O}$ be an open subset of $\mathbb{C}$, and $f \in \mathbf{Hol}\,\mathcal{O}$. Consider $z_0 \in \mathcal{O}$ and $\delta > 0$ such that $\overline{B_\delta(z_0)} \in \mathcal{O}$. Then, for $C = C_r(z_0)$, we have $f(z) = \sum a_n(z - z_0)^n$, where

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

Proof. By Cauchy integral formula for a circular contour, $\forall\, z \in B_r(z_0)$,

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

Now, the following series converges uniformly for $\zeta \in C \setminus \{z_0\}$.

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)}$$

$$= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n$$

Then, as Reimann integration behaves well under uniform convergence over compact domains, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right) d\zeta \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(\zeta)}{(\zeta - z_0^{n+1})} d\zeta \\ &= \sum_{n=0}^{\infty} a_n (z - z_0)^n \end{aligned}$$

Thus f is analytic over $\mathcal{O}$, with

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \tag{7.1}$$

□

The equations (7.1) are known as Cauchy integral formulae. This theorem sets apart complex analysis from our familiar terrain of real analysis - and has many striking consequences. We will see many applications of this wonderful theorem in the next lecture. For now, we have the simple corollary:

Corollary

For $\mathcal{O}$ an open subset of $\mathbb{C}$, $f \in \mathbf{Hol}(\mathcal{O})$ if and only if f is analytic on $\mathcal{O}$.

Lecture 8

8.1 Consequences of analyticity of holomorphic functions

Recall the main result from the last lecture: if $f \in \mathsf{Hol}(\mathcal{O})$, where $\mathcal{O}$ is an open subset of $\mathbb{C}$, and $\overline{B_r(z_0)} \subseteq \mathcal{O}$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \forall\, z \in B_r(z_0)$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, \mathrm{d}\zeta\, .$$

We now use focus on some important consequences of this result.

Theorem 8.1.1 (Cauchy's Inequality) Let $f \in \mathsf{Hol}(\mathcal{O})$ and $\overline{B_r(z_0)} \subseteq \mathcal{O}$. Then, for all $n \geq 1$,

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{r^n} \|f\|_{C_r(z_0)}\, ,$$

where we define

$$\|f\|_X = \sup_{z \in X} |f|.$$

Proof. We have,

$$\begin{aligned} \left| f^{(n)}(z_0) \right| &= \frac{n!}{2\pi} \left| \oint_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, \mathrm{d}\zeta \right| \\ &\leq \frac{n!}{2\pi} \sup_{z \in C_r(z_0)} \left| \frac{f(z)}{(z - z_0)^{n+1}} \right| \ell(C_r(z_0)) && \text{(triangle inequality)} \end{aligned}$$

We now use the fact that $|z - z_0| = r$ for $z \in C_r(z_0)$ and $\ell(C_r(z_0)) = 2\pi r$ to get

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{r^n} \|f\|_{C_r(z_0)}$$

as was required. □

Theorem 8.1.2 (Liouville's theorem) Let f be a bounded and entire function, i.e, $f \in \mathsf{Hol}(\mathbb{C})$. Then, f is constant.

Proof. We use Theorem 8.1.1 for f'. Fix $z_0 \in \mathbb{C}$ and $r > 0$. As $\overline{B_r(z_0)} \subseteq \mathbb{C}$, we have

$$|f'(z_0)| \leq \frac{1}{r} \|f\|_{C_r(z_0)} \leq \frac{\|f\|_\infty}{r}.$$

As r is an arbitrary positive real, we get $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$. Hence, f is constant. □

Remarks:

(1) As $\sin z, \cos z$ are non-constant entire functions, they are not bounded on $\mathbb{C}$.

(2) The theorem is false for $\mathbb{R}^n$, as there are analytic functions (e.g, $\sin x$) that are bounded and non-constant.

We now give a rather slick proof of one of the most useful and ubiquitous results in all of mathematics, the fact that $\mathbb{C}$ is algebraically closed.

> **Theorem 8.1.3 (Fundamental Theorem of Algebra)** Let $p \in \mathbb{C}[z]$ be non-constant. Then, there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Proof. Suppose no such z_0 exists. Then, $\frac{1}{p}$ is an entire function. If $p(z) = \sum_{n=0}^{d} a_n z^n$ for complex numbers a_n, then we get

$$\frac{1}{p(z)} = \frac{1}{z^d} \frac{1}{\sum_{n=0}^{d} a_n z^{n-d}} \xrightarrow{z \to \infty} 0$$

Therefore, $\frac{1}{p}$ is bounded for $|z| > R$, for some $R > 0$, and as $\frac{1}{p}$ is entire and in particular continuous, it attains a finite supremum on $\overline{B_R(0)}$. So, $\frac{1}{p}$ is a bounded entire function, and hence must be constant by Theorem 8.1.2. This contradicts the assumption that p is non-constant, and hence, some root z_0 of p exists in $\mathbb{C}$. $\square$

Recall Theorem 4.1 that if $f \in C(\mathcal{U})$ for some open convex set $\mathcal{U}$ and $\oint_{\partial\Delta} f = 0$ for all $\Delta \subseteq \mathcal{U}$, then there exists $g \in \mathsf{Hol}(\mathcal{U})$ such that $g' = f$. One corollary of this is that any such function f must be holomorphic on the set $\mathcal{U}$. A stronger version of this result is the following.

> **Theorem 8.1.4 (Morera's theorem)** Let $\mathcal{O} \subseteq \mathbb{C}$ be any open set, and $f \in C(\mathcal{O})$. Then, $f \in \mathsf{Hol}(\mathcal{U})$ if and only if $\oint_{\partial\Delta} f = 0$ for all $\Delta \subseteq \mathcal{O}$.

Proof. If $f \in \mathsf{Hol}(\mathcal{O})$, we get $\oint_{\partial\Delta} f = 0$ for all $\Delta \subseteq \mathcal{O}$ by Cauchy's integral theorem.

Conversely, suppose that $\oint_{\partial\Delta} f = 0$ for all $\Delta \subseteq \mathcal{O}$. Then, as any $\Delta \subseteq \mathcal{O}$ is a closed convex set, we get an open convex set $\mathcal{U}$ containing Δ which is also contained in $\mathcal{O}$. Over $\mathcal{U}$, we apply Theorem 4.1 and get a local primitive of f. In particular, $f \in \mathsf{Hol}(\mathcal{U})$. Therefore, as f is holomorphic at each point of $\mathcal{O}$, we get $f \in \mathsf{Hol}(\mathcal{O})$ and so we are done. $\square$

> **Theorem 8.1.5** Let $\{f_n\}_{\mathbb{N}} \subseteq \mathsf{Hol}(\mathcal{O})$. Suppose $f_n \to f$ uniformly. Then, $f \in \mathsf{Hol}(\mathcal{O})$.

Remarks:

(1) We define $f_n \to f$ uniformly on an open set, if the convergence is uniform on any compact subset of the open set.

(2) The result fails miserably for $\mathbb{R}$! For example, the Weierstrass approximation theorem says $\overline{\mathbb{R}[x]} \simeq C([0,1], \mathbb{R})$, that is, *any* continuous function can be approximated uniformly by polynomial functions, which are not just smooth but in fact their derivatives vanish after some finite order.

Proof. By uniform convergence, $f \in C(\mathcal{O})$. Fix $\Delta \subseteq \mathcal{O}$. As $f_n \in \mathsf{Hol}(\mathcal{O})$, we get by Morera's theorem that $\oint_{\partial\Delta} f = 0$. But,

$$\oint_{\partial\Delta} f = \oint_{\partial\Delta} \lim_{n\to\infty} f_n = \lim_{n\to\infty} \oint_{\partial\Delta} f_n = 0$$

and so by Morera's theorem again, $f \in \mathsf{Hol}(\mathcal{O})$. $\square$

Theorem 8.1.6 Let the same assumptions hold as Theorem 8.1.5. Then, for all $k \geq 1$,

$$f_n^{(k)} \to f^{(k)}$$

uniformly on $\mathcal{O}$.

Proof. It is clearly enough to show the result for $k = 1$, that is, $f_n' \to f'$ uniformly. Further, it is enough to show that $f_n' \to f'$ uniformly on the closed discs $\overline{B_r(z_0)} \subseteq \mathcal{O}$. Fix r, z_0 and δ small such that

$$\overline{B_r(z_0)} \subseteq \overline{B_{r+\delta}(z_0)} \subseteq \mathcal{O}.$$

Consider $z \in \overline{B_r(z_0)}$ and let C be a circle of radius R centered at z. Assume $\frac{\delta}{2} < R < \delta$ so that $C \subseteq \mathcal{O}$. By Theorem 8.1.1,

$$\begin{aligned}
|(f_n - f)'(z)| &\leq \frac{1}{R}\|f_n - f\|_C \\
\implies |(f_n - f)'(z)| &< \frac{1}{\delta}\|f_n - f\|_C \\
\implies |(f_n - f)'(z)| &< \frac{1}{\delta}\|f_n - f\|_{\overline{B_{r+\delta}(z_0)}} \\
\implies \|f_n' - f'\|_{\overline{B_r(z_0)}} &\leq \frac{1}{\delta}\|f_n - f\|_{\overline{B_{r+\delta}(z_0)}}
\end{aligned}$$

As $f_n \to f$ uniformly, we get that $f_n' \to f'$ uniformly from the above inequality. □

8.2 Zeroes of Analytic Functions

Let $\mathcal{O}$ be an open subset of $\mathbb{C}$, $f \in \mathsf{Hol}(\mathcal{O})$. We define the set of zeroes of f,

$$\mathcal{Z}(f) = \{z \in \mathcal{O} \mid f(z) = 0\}$$

Let $z_0 \in \mathcal{Z}(f)$. Consider the power series of f near z_0,

$$f(z) = \sum_{n \geq m} a_n (z - z_0)^n$$

We have the following two cases:

(i) There exists $m_0 \in \mathbb{N}$ such that $a_n = 0$ for all $n < m_0$ and $a_{m_0} \neq 0$.

Definition 8.2.1 ▶ Order of a zero

In this case, we define that z is a zero of f of (finite) order m_0, and write $\mathsf{Ord}(f; z_0) = m_0$.

We have in this case, for $z \in B_r(z_0)$,

$$f(z) = (z - z_0)^{m_0} g(z),$$

where $g(z_0) \neq 0$ and $g \in \mathsf{Hol}(B_r(z_0))$. As $g \in C(B_r(z_0))$ in particular, there is $r' \leq r$ such that $g(z) \neq 0$ for all $z \in B_{r'}(z_0)$. Therefore, $f(z) \neq 0$ for all $z \in B_{r'(z_0)} \setminus \{z_0\}$. We have thus proved the following theorem:

Theorem 8.2.1 Let $f \in \mathsf{Hol}(\mathcal{O})$ for $\mathcal{O}$ a domain. If $z_0 \in \mathcal{Z}(f)$ is a finite order zero of f, z_0 is an isolated point of $\mathcal{Z}(f)$.

(ii) $a_n = 0$ for all $n \geq 0$.

Definition 8.2.2

In this case, we define that z is a zero of f of infinite order.

We have in this case, $f \equiv 0$ on $B_r(z_0)$ for some $r > 0$. This proves the following theorem.

Theorem 8.2.2 The set

$$\widetilde{\mathcal{O}} = \{z \in \mathcal{Z}(f) \mid z \text{ an infinite zero of } f\}$$

is open.

The following theorem is an easy consequence of the above results but it is a immensely useful result by itself.

Theorem 8.2.3 (Nature of zeroes) Let $f \in \mathsf{Hol}(\mathcal{O})$ with $\mathcal{O}$ a domain in $\mathbb{C}$ and suppose f is not identically zero. Then all zeroes of f are of finite order and are isolated.

Proof. We know $\widetilde{\mathcal{O}}$ is open. We will show that it is also closed. Consider $z_0 \in \mathcal{O} \setminus \widetilde{\mathcal{O}}$. Then, there is $m \geq 0$ such that $f^{(m)}(z_0) \neq 0$. By continuity, there is $R > 0$ such that $f^{(m)}(z) \neq 0$ for all $z \in B_R(z_0) \subseteq \mathcal{O} \setminus \widetilde{\mathcal{O}}$. Hence, $\widetilde{\mathcal{O}}$ is a clopen set, and as it is not the full set $\mathcal{O}$, it must be empty. Therefore, all zeroes of f are of finite order, and they are isolated by the above results. □

Corollary

Let $f \in \mathsf{Hol}(\mathcal{O})$, $\mathcal{O}$ a domain. If $\mathcal{Z}(f)$ has a limit point in $\mathcal{O}$, then $f \equiv 0$.

The above result can be restated as follows:

Theorem 8.2.4 (Identity theorem) Suppose $f, g \in \mathsf{Hol}(\mathcal{O})$, $\mathcal{O}$ a domain. If $f = g$ on $X \subseteq \mathcal{O}$ which has a limit point in $\mathcal{O}$, then $f = g$ on all of $\mathcal{O}$.

Lecture 9

Recall in last lecture we proved '**Identity theorem**'. If $f \in \mathbf{Hol}(\mathcal{O})$ and if the zero set $Z(f)$ has limit point, then f is identically 0 over $\mathcal{O}$. For example,

Example- $f(z) = \sin \frac{1+z}{1-z}$, $z \in \mathbb{D}$, it is 0 when ever, $(1+z)/(1-z) = 2n\pi$ for integer n, i.e. $z = \frac{n\pi - 1}{n\pi + 1}$. This is a countable set and this sequence, do not have any limit point in $\mathbb{D}$.

> **Theorem 9.0.5 (Cauchy Integral theorem for simply connected domain)** Let $f \in \mathbf{Hol}(\mathcal{O})$. Then for any piece-wise smooth and closed curve γ,
>
> $$\oint_\gamma f(z)\, dz = 0$$

Proof. We know f is C^1 as it is holomorphic. We can use Green's theorem to get,

$$\oint_\gamma f\, dz = 2\int_{\Sigma_\gamma} \bar{\partial} f\, dx\, dy$$

Since $f \in \mathbf{Hol}(\mathcal{O})$ we have, $\bar{\partial} f = 0$ and thus the integral is zero. ■

REMARK : The roots of holomorphic functions are in some sense 'equivalent' to roots of polynomials.

> **Theorem 9.0.6 (Maximum Modulus Principle ;MMP)** Let, $f \in \mathbf{Hol}(\mathcal{O})$, where $\mathcal{O}$ is domain and $|f(z)| \leq |f(\alpha)|$ for some $\alpha \in \mathcal{O}$. Then $f \equiv$ constant.

Proof. Let us consider the set, $C = \{z \in \mathcal{O} : |f(z)| = |f(\alpha)|\}$. Since, this set is non-empty it is a closed set. Now, we will prove this is open as well, then by connectedness of $\mathcal{O}$ we can say, C is the whole set, and hence $|f|$ is constant over this domain $\Rightarrow f$ is constant.

Claim- *C is open, i.e. ever point is an interior point.*

Proof. Fix $z_0 \in C$, then there exists $R > 0$ such that $B_r(z_0) \subseteq \mathcal{O}$. Consider, $r < R$. Then,

$$|f(\alpha)| = |f(z)| = \left| \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right|$$

We now convert this in polar form. Define, $\zeta = z + re^{i\theta}, \theta \in [0.2\pi)$. Therefore we have,

$$\begin{aligned} |f(\alpha)| &= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(z + re^{i\theta})}{re^{i\theta}} re^{i\theta} i\, d\theta \right| \\ &= \frac{1}{2\pi} \left| \int_0^{2\pi} f(z + re^{i\theta})\, d\theta \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(z + re^{i\theta}) \right| d\theta \end{aligned}$$

But, $|f(\alpha)| \geq |f(z)|$ for all $z \in \mathcal{O}$. Thus, $\frac{1}{2\pi}\int_0^{2\pi}\left(|f(\alpha)| - \left|f(z+re^{i\theta})\right|\right)d\theta \leq 0$. From the above inequality and using continiuity we get $|f(\alpha)| = \left|f(z+re^{i\theta})\right|$. As this equality holds for all $r < R$ we can say this equality holds over $B_R(z_0)$. So, $B_R(z_0) \subseteq C$. Thus C is open. With this we have finished the proof of Claim as well as the theorem. ■

Corollary. *Let, $f \in \mathbf{Hol}(\mathcal{O})$ and in this open set $f(z) \neq 0$ for all z. Suppose, $|f(z)| \geq |f(\alpha)|$, for all z. Then f is constant on this domain.* [For proof just use MMP on $1/f$]

> **Theorem 9.0.7 (Open Mapping Theorem)** Let f is a holomorphic function on $\mathcal{O}$ and let it be a non-constant function. Then f is an open map.

Proof. (Caratheodory) Let, $\tilde{\mathcal{O}} \subseteq \mathcal{O}$ open. To prove $\mathcal{O}$ is open pick $\alpha \in \tilde{\mathcal{O}}$ and WLOG $f(\alpha) = 0$ contained in $f(\tilde{\mathcal{O}})$.

Claim- *There exist $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq f(\tilde{\mathcal{O}})$ for some $\varepsilon > 0$.*

Proof. As $f \neq$ constant, there exist a disc $\tilde{D}$ containing α such that $\tilde{D} \subseteq \tilde{\mathcal{O}}$ and $0 \notin f(\tilde{D} \setminus \{\alpha\})$. Evidently $f(\tilde{D}) \subseteq f(\tilde{\mathcal{O}})$. Thus enough to prove that for any such disc, there exist $B_\varepsilon(0) \subseteq f(\tilde{D})$. There exists a circle C centered at α such that $C \subseteq \tilde{D} \setminus \alpha$. Set, $\varepsilon := 1/2 \inf\{|f(z)| : z \in C\} > 0$. Set, $D = \Sigma_C$ enough to show that $B_\varepsilon(0) \subseteq D$. Now, fix $w \in B_\varepsilon(0)$ (in other words $|w| < \varepsilon$). Define $\eta(z) := f(z) - w$, for all z in the set D. Enough to prove, η has zero in D. We know, $\eta \in \mathbf{Hol}(D \subseteq$ domain). Now $|\eta(\alpha)| < \varepsilon$. Also, $z \in C$, $|\eta(z)| \geq |f(z)| - |w| > \varepsilon$. The MMP states that maxima or minima must occur at boundary. This is a contradiction !! The η has a zero in the set D, which completes the proof. ■

Special lecture -2

Recall the Bloch's theorem. We will look at few corollary of this theorem.

COROLLARY. *If f is holomorphic in domain G, $f'(c) \neq 0$ means $f(G)$ contains a disc of radii*

$$\frac{1}{12}s|f'(c)|$$

Towards little Picard

> Lemma 14.0.1
>
> Let, $G \subseteq \mathbb{C}$ simply-connected domain, let $f \in \mathbf{Hol}(G)$ and it's image con't contain $-1, 1$, then $f = \cos F$ for some holomorphic f.

Proposition Let $G \subseteq \mathbb{C}$ be a simply connected domain, $f(G)$ don't contain $0, 1$ then there exist $g \in \mathbf{Hol}(G)$ such that,

$$f(z) = \frac{1 + \cos(\cos \pi g)}{2}$$

and image of G does not contain any disc of radius 1.

If we take f to be as the thing given above, since f misses $0, 1$, $2f - 1$ will miss $-1, 1$. Now consider a set,

$$\mathbf{A} = \left\{ m \pm i \frac{\log\left(n + \sqrt{n^2 - 2}\right)}{\pi} : n \geq 2, m \in Z \right\}$$

This set will intersect disc of radius 1. But we will show $g(G) \cap \mathbf{A} = \emptyset$. Take $a \in \mathbf{A}$ and $a = m \pm i \frac{\log\left(n+\sqrt{n^2-2}\right)}{\pi}$ so we will have

$$\cos \pi a = (-1)^m n$$
$$f(a) = \frac{1 + (-1)^n}{2}$$

And hence it don't contain any disc of radius 1. ■

COROLLARY. *f is an entire function that does not fix any point then $f \circ f$ has a fixed point unless $f(z) = z + b$*

Proof. Consider the function, $g(z) = \frac{f \circ f(z) - z}{f(z) - z}$ misses 0 and 1, hence by Picard theorem we can say it is a constant function. Thus,

$$f(f(z)) - z = c(f(z) - 1)$$
$$\Rightarrow f'(f(z))f'(z) - 1 = c(f'(z) - 1)$$
$$\Rightarrow f'(z)(f'(f(z)) - c) = 1 - c$$

(complete the proof)

COROLLARY. *If f and g are entire and $f^n + g^n = 1$ then f, g are either constant or they have common poles.*
Proof.

Proof of Bloch's Theorem

Lemma 1

Let, $G \subseteq \mathbb{C}$ bounded and $f : G \to \mathbb{C}$ continuous suppose $f|_G$ is open. Let, $a \in G$ be such that $s = \min_{s \in \partial G} |f(z) - f(a)|$ then $f(G) \subseteq B(f(a), s)$
Proof. It is an exercise from Topology.

Lemma 2

Let, $V = B(a, r)$ and let $f \in \mathbf{Hol}(V)$ be non-constant. If $\|f'\|_V \leq 2|f'(a)|$, then

$$f(V) \supseteq B(f(a), R)$$

where $R = (3 - 2\sqrt{2})r|f(a)|$.

Proof. Consider the following function

$$A(z) = f(z) - f'(0)z = \int_{[0,z]} (f'(w) - f'(z))dw$$

Using Cauchy Integral formula we can say,

$$f'(v) - f'(0) = \frac{v}{2\pi i} \oint_{\partial V} \frac{f'(w)}{w(w - v)} dw$$

Now we have,

$$\begin{aligned} |A(z)| &\leq \int_0^1 |f'(zt) - f'(0)||z|\, dt \\ &\leq \int_0^1 \frac{|zt|\|f\|_V}{r - |zt|} |z|\, dt \\ &\leq \frac{|z|^2}{(r - |z|)} \|f'\|_V \end{aligned}$$

This $0 < \rho < r$, for $|z| = \rho$, $|f(z)| \geq \left(\rho - \frac{\rho^2}{r - \rho}\right)|f'(0)|$

Main Proof

Let, $f \in \mathbf{Hol}(E)$ it means $f(E) \supseteq B(f(a), (3/2 - 2\sqrt{2})M)$. Note, $|f'(z)|(1 - |z|)$ is continuous on $\bar{E}$. Suppose $p \in E$ be the maxima point of the above function. And let M be the maximum then $(3/2 - \sqrt{2})M > 1/12|f'(0)|$. Let, $t = \frac{1 - |p|}{2}$, this means $M = 2t|f'(p)|$ also $B(p, t) \subseteq E$ and $1 - |z| \geq t$ (via triangle inequality)

Lecture 18

Recall last day we talked about 'winding number' and it is a continuous function from $\mathbb{C}$ to $\mathbb{Z}$ is continuous. Today we will prove two big theorem, **Argument prnciple** and **Rouche's theorem**.

> **Theorem 18.0.8 (Argument Principle)** Let $\mathcal{O} \subseteq \mathbb{C}$ open, γ be a closed simple curve such that $\gamma \cup \Sigma\gamma \subseteq \mathcal{O}$. Suppose f is a meromorphic function. Then
>
> $$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = Z\{\#\text{ zeroes of f in }\Sigma\gamma\} - P\{\#\text{ poles of f in }\Sigma\gamma\}$$

Index