Lecture 1

1.1 Introduction

Our objective is to study functions $f : \mathbb{C} \to \mathbb{C}$. We know that as metric spaces $\mathbb{C}$ and $\mathbb{R}^2$ are isometric, with the natural map $(x, y) \mapsto x + iy$ being an isometry, but then what is the difference between analysis in $\mathbb{R}^2$ and analysis in $\mathbb{C}$? The difference arises because $\mathbb{C}$ is a field while $\mathbb{R}^2$ is not a field, thus we have a notion of multiplication and division in the complex plane.
Before going into further details we recall some of the obvious observations that one can make,

1. (Triangle inequality). $||z_1| - |z_2|| \leq |z_1 - z_2|$.
2. $|z| \geq \max\{|x|, |y|\}$ where $z = x + iy$.
3. $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is a Cauchy sequence if and only if $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^2$ is a Cauchy sequence, which is equivalent to $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are Cauchy sequences.
4. Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Then f is continuous (or limit exists) at a point $z_0 = x_0 + iy_0$ if and only if $f : \mathbb{R}^2 \to \mathbb{R}^2$ viewed as a function from the real plane to the real plane is continuous (or limit exists) at (x_0, y_0).

We haven't yet clarified how the analysis of $\mathbb{C}$ differs from the analysis of $\mathbb{R}^2$, the fact that $\mathbb{C}$ is a field gives us that $\frac{f(z)-f(z_0)}{z-z_0} \in \mathbb{C}$ for all $z \neq z_0$. Thus we can define the derivative of $f : \mathbb{C} \to C$ at z_0 as the complex number obtained by taking the limit $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ (provided the limit exists). Thus the derivative of $f : \mathbb{C} \to C$ at $z_0 \in \mathbb{C}$ is a complex number, while the derivative of $f : \mathbb{R}^2 \to \mathbb{R}^2$ (that is, the total derivative) is a 2×2 matrix.
This raises the following question let $f = u + iv$, then if we view $f = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$ we know

$$J_f(x_0, y_0) = Df(x_0, y_0) = \begin{bmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{bmatrix},$$

is there any relation between $f'(z_0)$ (if it exists) and $J_f(x_0, y_0)$?
Homewok. (another representation of $\mathbb{C}$). Let

$$M = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \mid x, y \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R}) \tag{1.1}$$

Show that M is a field under matrix multiplication and is in fact isomorphic to $\mathbb{C}$.

The above assignment suggests there must be some representation of $f'(z_0)$ in terms of the Jacobian matrix $J_f(x_0, y_0)$, indeed there is some relation which we will discuss in a while.
Notation. $B_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$.

Definition 1.1.1 ▶ Holomorphic Functions.

Let $\mathcal{O}$ be an open subset of $\mathbb{C}$, and let $f : \mathcal{O} \to \mathbb{C}$ be a function and $z_0 \in \mathcal{O}$. We say that f is

$\mathbb{C}$-differentiable at z_0 or **holomorphic at** z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \text{ exists.}$$

And we will say f is **holomorphic on** $\mathcal{O}$ if f is holomorphic at every point $z \in \mathcal{O}$. We will denote by

$$\mathbf{Hol}(\mathcal{O}) = \{f : \mathcal{O} \to \mathbb{C} \mid f \text{ is holomorphic}\}.$$

Then $\mathbf{Hol}(\mathcal{O})$ forms an algebra over $\mathbb{C}$.

Lemma 1.1.1 (Some Immediate Observations.)

Let $f, g : \mathcal{O} \to \mathbb{C}$ be holomorphic at z_0, then

1. f is continuous at z_0.
2. $(\alpha f + g)'(z_0) = \alpha f'(z_0) + g'(z_0)$ for all $\alpha \in \mathbb{C}$.

Example 1.1.1

Some examples of holomorphic functions are $f(z) = z$, $f(z) = \text{constant}$ and $f(z) = z^2$, while $f(z) = \bar{z}$ is not a holomorphic function. Note that in $\mathbb{R}^2$ the function $f(z) = \bar{z}$ corresponds to the function $f(u, v) = (u, -v)$. But then we get that $Df(u, v) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \notin M$ (where M is defined in equation 1.1). If we now consider the function $f(z) = z^2$, then in $\mathbb{R}^2$ it corresponds to the function $f(x, y) = (x^2 - y^2, 2xy)$ then $J_f(x, y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix} \in M$.

The above example gives us the motivation to answer the problem we had raised earlier: *how are the complex derivative and the Jacobian matrix related?*

1.2 Holomorphic versus Differentiable Functions.

For this discussion we will let $f = u + iv : \mathcal{O} \to \mathbb{C}$, let $z_0 = x_0 + iy_0$. Suppose f is holomorphic at z_0, and let $\alpha = a + ib = f'(z_0)$. We then define the function for all $z \in B_r(z_0)$

$$\begin{aligned} R(z) &= f(z) - f(z_0) - \alpha(z - z_0) \\ &= \underbrace{[u(z) - u(z_0) - a(x - x_0) + b(y - y_0)]}_{R_1(z)} + i \underbrace{[v(z) - v(z_0) - b(x - x_0) - a(y - y_0)]}_{R_2(z)}. \end{aligned}$$

Now recall that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at (x_0, y_0) if and only if

$$\frac{R(x, y)}{\|(x, y) - (x_0, y_0)\|} \to 0 \text{ as } (x, y) \to (x_0, y_0).$$

But we have

$$\frac{R(z)}{|z - z_0|} = \frac{R_1(z)}{|z - z_0|} + i \frac{R(z)}{|z - z_0|},$$

and we also know that f is holomorphic at z_0 hence we get that

$$\lim_{z \to z_0} \frac{R(z)}{|z - z_0|} = 0 \iff \lim_{z \to z_0} \frac{R_1(z)}{|z - z_0|} = \lim_{z \to z_0} \frac{R_2(z)}{|z - z_0|} = 0.$$

Thus it is equivalent to saying that $u, v : \mathcal{O} \to \mathbb{R}$ are differentiable at (x_0, y_0) and we further have

$$\begin{aligned} a &= u_x = v_y \\ b &= v_x = -u_y. \end{aligned}$$

> **Theorem 1.2.1 (Cauchy Riemann Equations)** Let $f := u + iv : \mathcal{O} \to \mathbb{C}$ be a function and $z_0 \in \mathcal{O}$. Then f is holomorphic at z_0 if and only if $u, v : \mathcal{O} \to \mathbb{R}$ is differentiable at z_0 and $u_x = v_y$ and $u_y = -v_x$. These are called the Cauchy Riemann Equations, thus we have
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> $$u_x = v_y,\ u_y = -v_x \quad \text{and} \quad f'(z_0) = u_x(z_0) + iv_x(z_0).$$