Lecture 4

4.1 Cauchy Integral Theorem

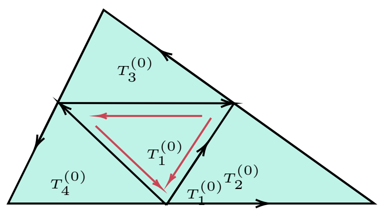
Last day we have seen, if a function have primitive over a simply connected domain, its' integral over the boundary of that region is 0. We also noted for a function $\frac{1}{z}$, integration (countour integral) over $\partial B_r(0)$ is not zero. So, it can't have a primitive over the region $\mathbb{C} \setminus \{0\}$. The function is Holomorphic on any open set do not contain 0. So the natural question arises,'Is it possible that, this function has primitive in some open set do not contain 0?' To answer this question we need 'Cauchy Integral Theorem', which is stated as following.

> **Theorem 4.1.1 (Cauchy(Goursat) Integral Theorem)** Let, $f \in \mathbf{Hol}(\mathcal{O})$, where $\mathcal{O}$ is simply connected domain. Then, $\int_\gamma f dz = 0$ for all piece-wise smooth closed curve $\gamma \subset \mathcal{O}$.

To prove this we need to develop more techniques in complex analysis. Rather, we will prove a weak version of the above theorem.

> **Theorem 4.1.2 (Cauchy Integral Theorem Δ version)** Let, γ is a triangle (curve including sides of the triangle T) and $\text{int}(T) \cup T \subseteq \mathcal{O}$. Then for all $f \in \mathbf{Hol}(\mathcal{O})$,
>
> $$\int_T f dx = 0$$

Proof. Lat, T^0 b the curve T with anticlockwise direction. Take the middle point of each side and joint them to get 4 triangles (as shown in the picture) with an orientation (shown in the picture). Call these triangles $T_j^{(0)}$ for $j = 1, \cdots, 4$.



Let, $I = \int_{T^{(0)}} f dz$. From the above partition we can say,

$$I = \int_{T^{(0)}} f dz = \sum_{j=1}^{4} \int_{T_j^{(0)}} f dz$$

$$|I| \leq \sum_{j=1}^{4} \left| \int_{T_j^{(0)}} f dz \right| \leq 4 \left| \int_{T_i^{(0)}} f dz \right|$$

The last inequality holds for some $i \in \{1, 2, 3, 4\}$. Now call this triangle $T_i^{(0)} := T^(1)$. We carry out the calculations for $i = 1, \cdots, n$ and obtain $\frac{1}{4^n}|I| \leq \left|\int_{T^{(n)}} f dz\right|$. Set, $\mathcal{T}^{(n)} := T^{(n)} \cup \text{int}(T^{(n)})$. These sets are compact for $n \in \mathbb{N}$. Thus, we have a chain of compact (closed subsets),

$$\mathcal{T}^{(0)} \supset \mathcal{T}^{(1)} \supset \cdots$$

with $\text{diam}\, \mathcal{T}^{(n)} = \frac{1}{2^n} \text{diam}\, \mathcal{T}^{(0)} = \frac{1}{2^n}($ length of largest side of $\mathcal{T}^{(0)})$. Using "**Cantor intersection**

theorem" we get,

$$\bigcap_{n=0}^{\infty} \mathcal{T}^{(n)} = \{z_0\} \text{ (singleton set)}$$

Using holomorphic property of f at z_0 we get, $f(z) = f(z_0) + f'(z_0)(z - z_0) + R(z)(z - z_0)$ holds in some open ball $B_\epsilon(z_0)$ around z_0, where $R(z)$ is continuous on that open ball with $\lim R(z) = 0$ as $z \to z_0$. Take n large enough so that $\mathcal{T}^{(n)} \subset B_\epsilon(z_0)$.

$$\begin{aligned}
\int_{T^{(n)}} f dz &= \int_{T^{(n)}} [f(z_0) + f'(z_0)(z - z_0) + R(z)(z - z_0)]dz \\
&= \int_{T^{(n)}} R(z)(z - z_0)dz \\
\implies \frac{1}{4^n}|I| &\leq \left| \int_{T^{(n)}} R(z)(z - z_0)dz \right|
\end{aligned}$$

Set, $\epsilon_n := \sup_{z \in T^{(n)}} |R(z)|$. Note that as $n \to \infty$, $\epsilon_n \to 0$. Also, for $z \in \mathcal{T}^{(n)}$ we have, $|z - z_0| \leq \operatorname{diam} \mathcal{T}^{(n)} = \frac{1}{2^n} d_0$ (here, d_0 is the length largest side of $T^{(n)}$). Using triangle inequality we get,

$$\begin{aligned}
\frac{1}{4^n}|I| \leq \left| \int_{T^{(n)}} R(z)(z - z_0)\, dz \right| &\leq \sup_{z \in T^{(n)}} |R(z)| \times \frac{d_0}{2^n} \times \operatorname{len}(T^{(n)}) \\
&= \frac{3d_0}{4^n} \cdot \epsilon_n \\
|I| \leq 3d\epsilon_n
\end{aligned}$$

Just by taking the limit $n \to \infty$ we have $I = 0$. ■

Corollary. *If R is a rectangle $R \subset \mathcal{O}$ and $f \in \mathbf{Hol}(\mathcal{O})$, then $\int_R f dz = 0$.*

Now we will look at the question, we concerned at the beginning. 'Does the primitive of $\frac{1}{z}$ exist in some domain $0 \notin B_\epsilon(z_0)$?' The answer is **Yes**! It should be given by $g(z) := \int_{\gamma_z} f\, dz$ where γ_z is a path from z_0 to z. The natural question should be to prove the well define-ness of the above g. We have to show, no matter what path b/w z and z_0 we choose, value of $g(z)$ must be same. If we have proven **main Cauchy theorem** then for any two path γ, η (that do not intersect each other except the end points) joining z and z_0, we will consider the concatenated loop $\gamma * \eta^{-1}$. Since the disc is simply-connected, the region bounded by the loop is also simply connected.Thus the theorem gives, $\int_{\gamma * \eta^{-1}} f\, dz = 0$ which gives $\int_\gamma f = -\int_{\eta^{-1}} f\, dz = \int_\eta f\, dz$. Thus, the function we defined is well-defined. Since we haven't proved the strong Cauchy theorem, we will answer the question with the help of following theorem.

> **Theorem 4.1.3** Let, $\mathcal{U} \subseteq \mathcal{O}$ open and convex. $f \in \operatorname{Cont}(\mathcal{U})$ and suppose $\int_{\partial\Delta} f\, dz = 0$ for all solid triangles $\Delta \subseteq \mathcal{U}$. Fix $z_0 \in \mathcal{U}$ and let $[z_0, z]$ be the line joining z_0 to z. Define,
>
> $$g(z) := \int_{[z_0, z]} f\, dz$$
>
> Then $g \in \mathbf{Hol}(\mathcal{U})$ and $g' = f$.

Proof. Fix $\tilde{z} \in \mathcal{U}$ and T_z be the triangle with sides $[z_0, \tilde{z}], [\tilde{z}, z], [z, z_0]$ (with this orientation). By the property of f we can say, $\int_{T_z} f\, dz = 0$. Expanding this integral we get,

$$\begin{aligned}
&\Rightarrow \int_{[z_0, \tilde{z}]} f\, dz + \int_{[\tilde{z}, z]} f\, dz + \int_{[z, z_0]} f\, dz = 0 \\
&\Rightarrow g(\tilde{z}) - g(z) = \int_{[z, \tilde{z}]} f\, dz \\
&\Rightarrow \frac{g(z) - g(\tilde{z})}{z - \tilde{z}} = \frac{1}{z - \tilde{z}} \int_{[z, \tilde{z}]} f\, dz
\end{aligned}$$

$$\Rightarrow \left| \frac{g(z) - g(\tilde{z})}{z - \tilde{z}} - f(\tilde{z}) \right| = \frac{1}{|z - \tilde{z}|} \left| \int_{[z,\tilde{z}]} f\, dz \right|$$

Using continuity of f we can say RHS of the above equation is $< \epsilon$ in some open nbd around $\tilde{z}$. Thus, g is holomorphic at $\tilde{z}$ with $f(\tilde{z}) = g'(\tilde{z})$, we can do this for any $\tilde{z} \in \mathcal{U}$ and hence we are done. ■

COROLLARY. *Let, $f \in \mathbf{Hol}(\mathcal{U})$ (with $\mathcal{U}$ is open and convex). Then there exist $g \in \mathbf{Hol}(\mathcal{U})$ such that $g' = f$ and $\int_\gamma f\, dz = 0$ for all smooth piece-wise smooth loop γ in $\mathcal{U}$.*

"One big theorem at a day"- *JDS*