Lecture 5

5.1 Stokes' and Green's Theorem

Before developing some more tools required to prove the Cauchy-Goursat theorem, we give some more motivation towards the result; what happens for C^1 functions. Recall Stokes's theorem for $\mathbb{R}^2$: Consider a simply connected region $\Omega \subseteq \mathbb{R}^2$ with piecewise smooth, simple and closed boundary $\partial\Omega$. If $f = P\,\mathrm{d}x + Q\,\mathrm{d}y$ is a C^1 $1-$form on an open set $U \supseteq \Omega \cup \partial\Omega$, then

$$\oint_{\partial\Omega} f = \iint_{\Omega} \mathrm{d}f\,.$$

Now, as $f = P\,\mathrm{d}x + Q\,\mathrm{d}y$, we get $\mathrm{d}f = \mathrm{d}P\,\mathrm{d}x + \mathrm{d}Q\,\mathrm{d}y = P_y\,\mathrm{d}y\,\mathrm{d}x + Q_x\,\mathrm{d}x\,\mathrm{d}y = (Q_x - P_y)\,\mathrm{d}x\,\mathrm{d}y$. Hence,

$$\oint_{\partial\Omega} f = \oint_{\partial\Omega} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y$$

which is nothing but the statement of Green's theorem. We now return to $\mathbb{C}$ and assume $f \in \mathsf{Hol}(\Omega) \cap C^1(\Omega)$, where Ω is a simply connected region as above. Let $\gamma \subseteq \Omega$ be a piecewise smooth, simple and closed curve. Then, by the discussion of lecture 3,

$$\oint_{\gamma} f\,\mathrm{d}z = \left(\oint_{\gamma} u\,\mathrm{d}x - v\,\mathrm{d}y\right) + i\left(\oint_{\gamma} v\,\mathrm{d}x + u\,\mathrm{d}y\right)$$

Using Stokes's theorem on the two integrals on the right we get, [Σ_γ is the capping surface of γ, in some lecture we have used the notion of capping surface as $\mathrm{int}\,(\gamma)$]

$$\oint_{\gamma} f\,\mathrm{d}z = \iint_{\Sigma\gamma} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y + i\iint_{\Sigma\gamma} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = 0$$

where the final equality follows from the Cauchy-Riemann equations. This is exactly the statement of the Cauchy-Goursat theorem! We now set to develop the tools needed to remove the C^1 restriction, that is, to show that holomorphic functions must be C^1 on "nice" domains.
Exercise: Suppose $f \in C^1(\Omega)$, where Ω is as above. If γ is a curve as above, show that

$$\oint_{\gamma} f\,\mathrm{d}z = 2i\iint_{\Sigma\gamma} \overline{\partial} f\,\mathrm{d}x\,\mathrm{d}y$$

holds for all such functions, without assuming holomorphicity.

Recall the result from lecture 3 that

$$\frac{1}{2\pi i}\oint_{C_r(0)} \frac{1}{z} = 1$$

for any $r > 0$. Changing variables, we get the equality

$$\frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{1}{z - z_0} = 1$$

for any $z_0 \in \mathbb{C}$. We now generalise this result to the following very useful theorem.

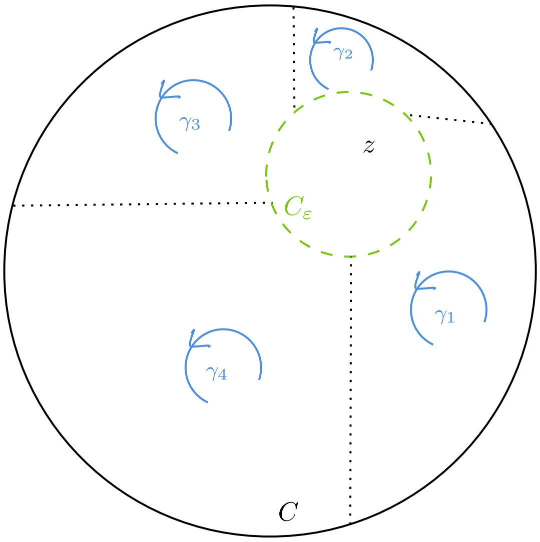
5.2 Cauchy Integral Formula

Theorem 5.2.1 (Cauchy Integral Formula) Let $f \in \mathsf{Hol}(\mathcal{O})$ and $C \subseteq \mathcal{O}$ be a circle such that $D = C \cup \Sigma C$ is in $\mathcal{O}$. Then, for all $z \in \Sigma C$,

$$\frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta = f(z).$$

Remark. Note that if $z \in \mathcal{O} \setminus D$, by Theorem 4.1,

$$\frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta = 0.$$



Proof. Fix $z \in \Sigma C$ and let $B_\varepsilon(z) \subseteq \Sigma C$, $C_\varepsilon = \partial B_\varepsilon(z)$. We claim,

$$\oint_C \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta = \oint_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

Introduce 4 "cuts" as shown by the dotted lines in the figure, and let the loops displayed be oriented anticlockwise. Then, $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 = C \cup (-C_\varepsilon)$. Further,

$$\zeta \mapsto \frac{f(\zeta)}{\zeta - z}$$

is holomorphic in $\Sigma\gamma_j$ for each j. As each of these sets $\Sigma\gamma_j$ can be covered by an open convex set $\Omega_j \subseteq \mathcal{O}$, we get by Theorem 4.1 that

$$\oint_{\gamma_j} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta = 0$$

for each j. Therefore, as integration over $\cup\gamma_j$ is simply the sum of each of these integrals, we get

$$\begin{aligned}
\oint_C \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta &= \oint_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \\
\implies \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - f(z) &= \frac{1}{2\pi i} \oint_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \frac{f(z)}{2\pi i} \oint_{C_\varepsilon} \frac{1}{\zeta - z} \, \mathrm{d}\zeta \\
\implies \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - f(z) &= \frac{1}{2\pi i} \oint_{C_\varepsilon} \frac{f(\zeta) - f(z)}{\zeta - z} \, \mathrm{d}\zeta
\end{aligned}$$

By the triangle inequality,

$$\left|\frac{1}{2\pi i}\oint_{C_\varepsilon}\frac{f(\zeta)-f(z)}{\zeta - z}\,\mathrm{d}\zeta\right| \le \frac{1}{2\pi}\sup_{\zeta\in C_\varepsilon}\left|\frac{f(\zeta)-f(z)}{\zeta - z}\right|\ell(C_\varepsilon) = \varepsilon\sup_{\zeta\in C_\varepsilon}\left|\frac{f(\zeta)-f(z)}{\zeta - z}\right|$$

As $\varepsilon \to 0$, the supremum goes to $|f'(z)|$ and so the right most quantity goes to 0. Therefore, in light of the equality above, we get

$$\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - f(z) = 0$$

as was to be shown. □