Lecture 8

8.1 Consequences of analyticity of holomorphic functions

Recall the main result from the last lecture: if $f \in \mathsf{Hol}(\mathcal{O})$, where $\mathcal{O}$ is an open subset of $\mathbb{C}$, and $\overline{B_r(z_0)} \subseteq \mathcal{O}$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad \forall\, z \in B_r(z_0)$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, \mathrm{d}\zeta\,.$$

We now use focus on some important consequences of this result.

> **Theorem 8.1.1 (Cauchy's Inequality)** Let $f \in \mathsf{Hol}(\mathcal{O})$ and $\overline{B_r(z_0)} \subseteq \mathcal{O}$. Then, for all $n \geq 1$,
>
> $$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{r^n} \|f\|_{C_r(z_0)}\,,$$
>
> where we define
>
> $$\|f\|_X = \sup_{z \in X} |f|.$$

Proof. We have,

$$\begin{aligned} \left| f^{(n)}(z_0) \right| &= \frac{n!}{2\pi} \left| \oint_{C_r(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, \mathrm{d}\zeta \right| \\ &\leq \frac{n!}{2\pi} \sup_{z \in C_r(z_0)} \left| \frac{f(z)}{(z-z_0)^{n+1}} \right| \ell(C_r(z_0)) && \text{(triangle inequality)} \end{aligned}$$

We now use the fact that $|z - z_0| = r$ for $z \in C_r(z_0)$ and $\ell(C_r(z_0)) = 2\pi r$ to get

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{r^n} \|f\|_{C_r(z_0)}$$

as was required. $\square$

> **Theorem 8.1.2 (Liouville's theorem)** Let f be a bounded and entire function, i.e, $f \in \mathsf{Hol}(\mathbb{C})$. Then, f is constant.

Proof. We use Theorem 8.1.1 for f'. Fix $z_0 \in \mathbb{C}$ and $r > 0$. As $\overline{B_r(z_0)} \subseteq \mathbb{C}$, we have

$$|f'(z_0)| \leq \frac{1}{r} \|f\|_{C_r(z_0)} \leq \frac{\|f\|_\infty}{r}.$$

As r is an arbitrary positive real, we get $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$. Hence, f is constant. $\square$

Remarks:

(1) As $\sin z, \cos z$ are non-constant entire functions, they are not bounded on $\mathbb{C}$.

(2) The theorem is false for $\mathbb{R}^n$, as there are analytic functions (e.g, $\sin x$) that are bounded and non-constant.

We now give a rather slick proof of one of the most useful and ubiquitous results in all of mathematics, the fact that $\mathbb{C}$ is algebraically closed.

Theorem 8.1.3 (Fundamental Theorem of Algebra) Let $p \in \mathbb{C}[z]$ be non-constant. Then, there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Proof. Suppose no such z_0 exists. Then, $\frac{1}{p}$ is an entire function. If $p(z) = \sum_{n=0}^{d} a_n z^n$ for complex numbers a_n, then we get

$$\frac{1}{p(z)} = \frac{1}{z^d} \frac{1}{\sum_{n=0}^{d} a_n z^{n-d}} \xrightarrow{z \to \infty} 0$$

Therefore, $\frac{1}{p}$ is bounded for $|z| > R$, for some $R > 0$, and as $\frac{1}{p}$ is entire and in particular continuous, it attains a finite supremum on $\overline{B_R(0)}$. So, $\frac{1}{p}$ is a bounded entire function, and hence must be constant by Theorem 8.1.2. This contradicts the assumption that p is non-constant, and hence, some root z_0 of p exists in $\mathbb{C}$. $\square$

Recall Theorem 4.1 that if $f \in C(\mathcal{U})$ for some open convex set $\mathcal{U}$ and $\oint_{\partial\Delta} f = 0$ for all $\Delta \subseteq \mathcal{U}$, then there exists $g \in \mathsf{Hol}(\mathcal{U})$ such that $g' = f$. One corollary of this is that any such function f must be holomorphic on the set $\mathcal{U}$. A stronger version of this result is the following.

Theorem 8.1.4 (Morera's theorem) Let $\mathcal{O} \subseteq \mathbb{C}$ be any open set, and $f \in C(\mathcal{O})$. Then, $f \in \mathsf{Hol}(\mathcal{U})$ if and only if $\oint_{\partial\Delta} f = 0$ for all $\Delta \subseteq \mathcal{O}$.

Proof. If $f \in \mathsf{Hol}(\mathcal{O})$, we get $\oint_{\partial\Delta} f = 0$ for all $\Delta \subseteq \mathcal{O}$ by Cauchy's integral theorem.

Conversely, suppose that $\oint_{\partial\Delta} f = 0$ for all $\Delta \subseteq \mathcal{O}$. Then, as any $\Delta \subseteq \mathcal{O}$ is a closed convex set, we get an open convex set $\mathcal{U}$ containing Δ which is also contained in $\mathcal{O}$. Over $\mathcal{U}$, we apply Theorem 4.1 and get a local primitive of f. In particular, $f \in \mathsf{Hol}(\mathcal{U})$. Therefore, as f is holomorphic at each point of $\mathcal{O}$, we get $f \in \mathsf{Hol}(\mathcal{O})$ and so we are done. $\square$

Theorem 8.1.5 Let $\{f_n\}_{\mathbb{N}} \subseteq \mathsf{Hol}(\mathcal{O})$. Suppose $f_n \to f$ uniformly. Then, $f \in \mathsf{Hol}(\mathcal{O})$.

Remarks:

(1) We define $f_n \to f$ uniformly on an open set, if the convergence is uniform on any compact subset of the open set.

(2) The result fails miserably for $\mathbb{R}$! For example, the Weierstrass approximation theorem says $\overline{\mathbb{R}[x]} \simeq C([0,1], \mathbb{R})$, that is, *any* continuous function can be approximated uniformly by polynomial functions, which are not just smooth but in fact their derivatives vanish after some finite order.

Proof. By uniform convergence, $f \in C(\mathcal{O})$. Fix $\Delta \subseteq \mathcal{O}$. As $f_n \in \mathsf{Hol}(\mathcal{O})$, we get by Morera's theorem that $\oint_{\partial\Delta} f = 0$. But,

$$\oint_{\partial\Delta} f = \oint_{\partial\Delta} \lim_{n \to \infty} f_n = \lim_{n \to \infty} \oint_{\partial\Delta} f_n = 0$$

and so by Morera's theorem again, $f \in \mathsf{Hol}(\mathcal{O})$. $\square$

Theorem 8.1.6 Let the same assumptions hold as Theorem 8.1.5. Then, for all $k \geq 1$,

$$f_n^{(k)} \to f^{(k)}$$

uniformly on $\mathcal{O}$.

Proof. It is clearly enough to show the result for $k = 1$, that is, $f_n' \to f'$ uniformly. Further, it is enough to show that $f_n' \to f'$ uniformly on the closed discs $\overline{B_r(z_0)} \subseteq \mathcal{O}$. Fix r, z_0 and δ small such that

$$\overline{B_r(z_0)} \subseteq \overline{B_{r+\delta}(z_0)} \subseteq \mathcal{O}.$$

Consider $z \in \overline{B_r(z_0)}$ and let C be a circle of radius R centered at z. Assume $\frac{\delta}{2} < R < \delta$ so that $C \subseteq \mathcal{O}$. By Theorem 8.1.1,

$$\begin{aligned}
&|(f_n - f)'(z)| \leq \frac{1}{R}\|f_n - f\|_C \\
\implies &|(f_n - f)'(z)| < \frac{1}{\delta}\|f_n - f\|_C \\
\implies &|(f_n - f)'(z)| < \frac{1}{\delta}\|f_n - f\|_{\overline{B_{r+\delta}(z_0)}} \\
\implies &\|f_n' - f'\|_{\overline{B_r(z_0)}} \leq \frac{1}{\delta}\|f_n - f\|_{\overline{B_{r+\delta}(z_0)}}
\end{aligned}$$

As $f_n \to f$ uniformly, we get that $f_n' \to f'$ uniformly from the above inequality. □

8.2 Zeroes of Analytic Functions

Let $\mathcal{O}$ be an open subset of $\mathbb{C}$, $f \in \mathsf{Hol}(\mathcal{O})$. We define the set of zeroes of f,

$$\mathcal{Z}(f) = \{z \in \mathcal{O} \mid f(z) = 0\}$$

Let $z_0 \in \mathcal{Z}(f)$. Consider the power series of f near z_0,

$$f(z) = \sum_{n \geq m} a_n (z - z_0)^n$$

We have the following two cases:

(i) There exists $m_0 \in \mathbb{N}$ such that $a_n = 0$ for all $n < m_0$ and $a_{m_0} \neq 0$.

Definition 8.2.1 ▶ Order of a zero

In this case, we define that z is a zero of f of (finite) order m_0, and write $\mathsf{Ord}(f; z_0) = m_0$.

We have in this case, for $z \in B_r(z_0)$,

$$f(z) = (z - z_0)^{m_0} g(z),$$

where $g(z_0) \neq 0$ and $g \in \mathsf{Hol}(B_r(z_0))$. As $g \in C(B_r(z_0))$ in particular, there is $r' \leq r$ such that $g(z) \neq 0$ for all $z \in B_{r'}(z_0)$. Therefore, $f(z) \neq 0$ for all $z \in B_{r'(z_0)} \setminus \{z_0\}$. We have thus proved the following theorem:

Theorem 8.2.1 Let $f \in \mathsf{Hol}(\mathcal{O})$ for $\mathcal{O}$ a domain. If $z_0 \in \mathcal{Z}(f)$ is a finite order zero of f, z_0 is an isolated point of $\mathcal{Z}(f)$.

(ii) $a_n = 0$ for all $n \geq 0$.

Definition 8.2.2

In this case, we define that z is a zero of f of infinite order.

We have in this case, $f \equiv 0$ on $B_r(z_0)$ for some $r > 0$. This proves the following theorem.

Theorem 8.2.2 The set

$$\widetilde{\mathcal{O}} = \{z \in \mathcal{Z}(f) \mid z \text{ an infinite zero of } f\}$$

is open.

The following theorem is an easy consequence of the above results but it is a immensely useful result by itself.

Theorem 8.2.3 (Nature of zeroes) Let $f \in \mathsf{Hol}(\mathcal{O})$ with $\mathcal{O}$ a domain in $\mathbb{C}$ and suppose f is not identically zero. Then all zeroes of f are of finite order and are isolated.

Proof. We know $\widetilde{\mathcal{O}}$ is open. We will show that it is also closed. Consider $z_0 \in \mathcal{O} \setminus \widetilde{\mathcal{O}}$. Then, there is $m \geq 0$ such that $f^{(m)}(z_0) \neq 0$. By continuity, there is $R > 0$ such that $f^{(m)}(z) \neq 0$ for all $z \in B_R(z_0) \subseteq \mathcal{O} \setminus \widetilde{\mathcal{O}}$. Hence, $\widetilde{\mathcal{O}}$ is a clopen set, and as it is not the full set $\mathcal{O}$, it must be empty. Therefore, all zeroes of f are of finite order, and they are isolated by the above results. $\square$

Corollary

Let $f \in \mathsf{Hol}(\mathcal{O})$, $\mathcal{O}$ a domain. If $\mathcal{Z}(f)$ has a limit point in $\mathcal{O}$, then $f \equiv 0$.

The above result can be restated as follows:

Theorem 8.2.4 (Identity theorem) Suppose $f, g \in \mathsf{Hol}(\mathcal{O})$, $\mathcal{O}$ a domain. If $f = g$ on $X \subseteq \mathcal{O}$ which has a limit point in $\mathcal{O}$, then $f = g$ on all of $\mathcal{O}$.