Lecture 9

Recall in last lecture we proved '**Identity theorem**'. If $f \in \mathbf{Hol}(\mathcal{O})$ and if the zero set $Z(f)$ has limit point, then f is identically 0 over $\mathcal{O}$. For example,

Example- $f(z) = \sin \frac{1+z}{1-z}$, $z \in \mathbb{D}$, it is 0 when ever, $(1+z)/(1-z) = 2n\pi$ for integer n, i.e. $z = \frac{n\pi - 1}{n\pi + 1}$. This is a countable set and this sequence, do not have any limit point in $\mathbb{D}$.

> **Theorem 9.0.1 (Cauchy Integral theorem for simply connected domain)** Let $f \in \mathbf{Hol}(\mathcal{O})$. Then for any piece-wise smooth and closed curve γ,
>
> $$\oint_\gamma f(z)\, dz = 0$$

Proof. We know f is C^1 as it is holomorphic. We can use Green's theorem to get,

$$\oint_\gamma f\, dz = 2 \int_{\Sigma_\gamma} \bar{\partial} f\, dx\, dy$$

Since $f \in \mathbf{Hol}(\mathcal{O})$ we have, $\bar{\partial} f = 0$ and thus the integral is zero. ∎

REMARK : The roots of holomorphic functions are in some sense 'equivalent' to roots of polynomials.

> **Theorem 9.0.2 (Maximum Modulus Principle ;MMP)** Let, $f \in \mathbf{Hol}(\mathcal{O})$, where $\mathcal{O}$ is domain and $|f(z)| \leq |f(\alpha)|$ for some $\alpha \in \mathcal{O}$. Then $f \equiv$ constant.

Proof. Let us consider the set, $C = \{z \in \mathcal{O} : |f(z)| = |f(\alpha)|\}$. Since, this set is non-empty it is a closed set. Now, we will prove this is open as well, then by connectedness of $\mathcal{O}$ we can say, C is the whole set, and hence $|f|$ is constant over this domain $\Rightarrow f$ is constant.

Claim- *C is open, i.e. ever point is an interior point.*

Proof. Fix $z_0 \in C$, then there exists $R > 0$ such that $B_r(z_0) \subseteq \mathcal{O}$. Consider, $r < R$. Then,

$$|f(\alpha)| = |f(z)| = \left| \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \right|$$

We now convert this in polar form. Define, $\zeta = z + re^{i\theta}, \theta \in [0.2\pi)$. Therefore we have,

$$\begin{aligned}
|f(\alpha)| &= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(z + re^{i\theta})}{re^{i\theta}} re^{i\theta} i\, d\theta \right| \\
&= \frac{1}{2\pi} \left| \int_0^{2\pi} f(z + re^{i\theta})\, d\theta \right| \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(z + re^{i\theta}) \right| d\theta
\end{aligned}$$

But, $|f(\alpha)| \geq |f(z)|$ for all $z \in \mathcal{O}$. Thus, $\frac{1}{2\pi} \int_0^{2\pi} \left(|f(\alpha)| - \left| f(z + re^{i\theta}) \right| \right) d\theta \leq 0$. From the above inequality and using continiuity we get $|f(\alpha)| = \left| f(z + re^{i\theta}) \right|$. As this equality holds for all $r < R$

we can say this equality holds over $B_R(z_0)$. So, $B_R(z_0) \subseteq C$. Thus C is open. With this we have finished the proof of Claim as well as the theorem. ■

COROLLARY. *Let, $f \in \mathbf{Hol}(\mathcal{O})$ and in this open set $f(z) \neq 0$ for all z. Suppose, $|f(z)| \geq |f(\alpha)|$, for all z. Then f is constant on this domain.* [For proof just use MMP on $1/f$]

Theorem 9.0.3 (Open Mapping Theorem) Let f is a holomorphic function on $\mathcal{O}$ and let it be a non-constant function. Then f is an open map.

Proof. (CARATHEODORY) Let, $\tilde{\mathcal{O}} \subseteq \mathcal{O}$ open. To prove $\mathcal{O}$ is open pick $\alpha \in \tilde{\mathcal{O}}$ and WLOG $f(\alpha) = 0$ contained in $f(\tilde{\mathcal{O}})$.

Claim- *There exist $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq f(\tilde{\mathcal{O}})$ for some $\varepsilon > 0$.*

Proof. As $f \neq$ constant, there exist a disc $\tilde{D}$ containing α such that $\tilde{D} \subseteq \tilde{\mathcal{O}}$ and $0 \notin f(\tilde{D} \setminus \{\alpha\})$. Evidently $f(\tilde{D}) \subseteq f(\tilde{\mathcal{O}})$. Thus enough to prove that for any such disc, there exist $B_\varepsilon(0) \subseteq f(\tilde{D})$. There exists a circle C centered at α such that $C \subseteq \tilde{D} \setminus \alpha$. Set, $\varepsilon := 1/2 \inf \{|f(z)| : z \in C\} > 0$. Set, $D = \Sigma_C$ enough to show that $B_\varepsilon(0) \subseteq D$. Now, fix $w \in B_\varepsilon(0)$ (in other words $|w| < \varepsilon$). Define $\eta(z) := f(z) - w$, for all z in the set D. Enough to prove, η has zero in D. We know, $\eta \in \mathbf{Hol}(D \subseteq$ domain $)$. Now $|\eta(\alpha)| < \varepsilon$. Also, $z \in C$, $|\eta(z)| \geq |f(z)| - |w| > \varepsilon$. The MMP states that maxima or minima must occur at boundary. This is a contradiction !! The η has a zero in the set D, which completes the proof. ■