The BGG Resolution

Lie Algebras Fall 2024

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Introduction

The construction of Cartan matrices and the Verma modules are two big steps in understanding the representation theory of finite-dimensional complex semisimple Lie algebras, the culmination being that if $V \in \text{irRep}$ has highest weight $\lambda \in \Lambda^+$, then V is the quotient of M_λ by its maximal submodule.

Introduction

The construction of Cartan matrices and the Verma modules are two big steps in understanding the representation theory of finite-dimensional complex semisimple Lie algebras, the culmination being that if $V \in \text{irRep}$ has highest weight $\lambda \in \Lambda^+$, then V is the quotient of M_λ by its maximal submodule.

The BGG Resolution is a further step, and describes V as the last term of a long exact sequence consisting of direct sums of Verma modules in all other places. This ties in classical representation theory with the modern techiques of Lie algebra cohomology.

Notation and Preliminaries

Notation

$\mathfrak{g}$ will be a finite dimensional semisimple Lie algebra over $\mathbb{C}$, with fixed triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \ .$$

Notation

- The set of simple roots is $\{\alpha_i\}_\Sigma$, and the set of fundamental weights is $\{\omega_i\}_\Sigma$. The weight lattice and the root lattice are then: $\Lambda = \mathbb{Z}\{\omega_i\}_\Sigma$ and $Q = \mathbb{Z}\{\alpha_i\}_\Sigma$.
- For $\lambda, \mu \in \Lambda, \lambda \geq \mu \iff \lambda - \mu \in Q^+$.
- The Verma module of weight λ is denoted as M_λ.
- The Weyl group is W, and we also set

$$W_k = \{w \in W \mid \ell(w) = k\}, k \geq 1$$

- The affine action of the Weyl group is

$$w \circ \lambda = w(\lambda + \varrho) - \varrho,$$

  where $\varrho = \sum_\Sigma \alpha = \frac{1}{2} \sum_{\Phi_+} \alpha$.
- If M is a $\mathfrak{g}-$module, the $\mu-$weight subspace is

$$M^\mu = \{v \in M \mid \mathfrak{h}v = \mu(\mathfrak{h})v\}$$

Category $\mathscr{O}$

Definition ▶ Category $\mathscr{O}$

Let $\mathscr{O}$ be the full subcategory of left $U\mathfrak{g}-$modules such that if M is in $\mathscr{O}$ then:

(i) M is $U\mathfrak{g}-$finitely generated.

(ii) M is $\mathfrak{h}-$semisimple, ie, it is a weight module.

(iii) M is locally $U\mathfrak{n}^+-$finite, ie,

$$\dim \operatorname{Span}_{U\mathfrak{n}^+}(v) < \infty, \forall\, v \in M.$$

The BGG resolution

Homs of Verma modules

Theorem (Verma)

For $\lambda, \mu \in \mathfrak{h}^*, \mathrm{Hom}_{\mathfrak{g}}(M_\lambda, M_\mu)$ is either 0 or 1 dimensional, and any non-zero morphism is injective. Moreover, if $\lambda \in \Lambda^+$,

$$\mathrm{Hom}_{\mathfrak{g}}\left(M_{w_1 \circ \lambda}, M_{w_2 \circ \lambda}\right) = \mathbb{C} \iff w_1 \geq w_2.$$

Recall that $w_1 \geq w_2$ means that we have a chain

$$w_1 \to u_1 \to \cdots \to u_{n-1} \to w_2,$$

where $u_k = s_{i_{k+1}} u_{k+1}$ for some simple reflection $s_{i_{k+1}}$.

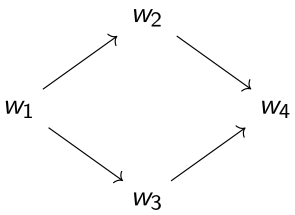
Homs of Verma modules

Hence, for $w_1 \geq w_2$, and $\lambda \in \Lambda^+$, there is a canonical embedding

$$\iota_{w_1 \to w_2} : M_{w_1 \circ \lambda} \hookrightarrow M_{w_2 \circ \lambda}.$$

Homs of Verma modules

We assemble the above data in a graph $\Gamma(W)$ as follows: let each element $w \in W$ be a vertex, and add a directed edge $w_1 \to w_2$ when $w_1 = sw_2$ for some simple reflection s. We call a tuple (w_1, w_2, w_3, w_4) a **square** if we have a subgraph



Homs of Verma modules

Theorem (BGG 10.3, 10.4)

For $w_1, w_4 \in W$ with $\ell(w_1) = \ell(w_4) + 2$, there are either zero or two vertices that fit into a square ending at w_1 and w_4. Moreover, to each arrow $w_1 \to w_2$ in $\Gamma(W)$ we can assign a sign $\operatorname{sgn}(w_1, w_2) = \pm 1$, such that

$$\prod_{\text{arrows in a square}} \operatorname{sgn}(w, w') = -1.$$

Statement of the Theorem

Fix $\lambda \in \Lambda^+$. We place a Verma module $M_{w \circ \lambda}$ at the vertex w, and grade the graph by $\ell(w)$. For each arrow $w_1 \to w_2$ we have a map

$$\operatorname{sgn}(w_1, w_2)\iota_{w_1 \to w_2} : M_{w_1 \circ \lambda} \to M_{w_2 \circ \lambda}.$$

We direct sum all the modules in the same grading, and get the differential by appropriately combining these maps.

Statement of the Theorem

Theorem (BGG Resolution)

For $V \in \mathrm{irRep}_{\mathrm{fd}}(\mathfrak{g})$ of highest weight $\lambda \in \Lambda^+$, there is a resolution

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\mathrm{d}_{k+1}} & \bigoplus_{w \in W_k} M_{w \circ \lambda} & \xrightarrow{\mathrm{d}_k} & \cdots & \xrightarrow{\mathrm{d}_2} & \bigoplus M_{s_i \circ \lambda} & \xrightarrow{\mathrm{d}_1} & M_\lambda \\
\uparrow {\scriptstyle \mathrm{d}_{|\Phi_+|}} & & & & & & & & \downarrow {\scriptstyle \mathrm{d}_0} \\
M_{w_0 \circ \lambda} & & & & & & & & V \\
\uparrow & & & & & & & & \downarrow \\
0 & & & & & & & & 0
\end{array}$$

where d_k is defined as $\mathrm{d}_k\big|_{M_{w \circ \lambda}} = (\mathrm{sgn}\,(w, w')\iota_{w \to w'})_{w' \in W_{k-1}}$, and $\mathrm{d}_0 : M_\lambda \twoheadrightarrow V$ is the canonical projection.

The Weyl Character formula

Weyl's Character Formula

Theorem

For $V \in \mathrm{irRep}_{\mathrm{fd}}(\mathfrak{g})$ of highest weight $\lambda \in \Lambda^+$, the character of V is given by

$$\chi_V = \sum_{w \in W} \operatorname{sgn}(w) e^{w \circ \lambda} \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^{-\alpha}},$$

where $\operatorname{sgn}(w) = (-1)^{\ell(w)}$.

Weyl's Character Formula

Lemma

For $\lambda \in \mathfrak{h}^*$, the Verma module M_λ has character

$$\chi_{M_\lambda} = e^\lambda \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^{-\alpha}}.$$

Proof of lemma

M_λ has weights $\mu \in \lambda - Q_+$, and each weight is finite dimensional. Further, $U\mathfrak{n}^- \simeq M_\lambda$ as vector spaces via $x \mapsto xv^\lambda$, and if $\alpha \in \Phi_+, f_\alpha v^\lambda \in M_\lambda^{\lambda-\alpha}$. By the PBW theorem, $U\mathfrak{n}^-$ has basis $\{\prod_\alpha f_\alpha^{n_\alpha}\}_{\alpha\in\Phi_+}$ and hence,

$$\dim M_\lambda^{\lambda-\delta} = \left|\left\{ \sum_{\alpha\in\Phi_+} n_\alpha \alpha \,\middle|\, \sum_{\alpha\in\Phi_+} n_\alpha\alpha = \delta \right\}\right|.$$

Proof of lemma

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$$\dim M_\lambda^{\lambda-\delta} = \left|\left\{ \sum_{\alpha\in\Phi_+} n_\alpha \alpha \,\middle|\, \sum_{\alpha\in\Phi_+} n_\alpha\alpha = \delta \right\}\right|$$

$$= [e^{-\delta}] \prod_{\alpha\in\Phi_+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots)$$

Proof of lemma

Hence, the character of the Verma module M_λ is

$$\chi_{M_\lambda} = \sum_{\mu \in \lambda - Q_+} \dim M_\lambda^\mu e^\mu = \sum_{\delta \in Q_+} \dim M_\lambda^{\lambda-\delta} e^{\lambda-\delta}$$

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Proof of lemma

Hence, the character of the Verma module M_λ is

$$\chi_{M_\lambda} = \sum_{\mu \in \lambda - Q_+} \dim M_\lambda^\mu e^\mu = \sum_{\delta \in Q_+} \dim M_\lambda^{\lambda-\delta} e^{\lambda-\delta}$$

$$= e^\lambda \sum_{\delta \in Q_+} e^{-\delta} \left([e^{-\delta}] \prod_{\alpha \in \Phi_+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots) \right)$$

$$= e^\lambda \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^{-\alpha}}$$

as required. ■

Proof of Weyl's formula

Consider a map $\varphi : M \to N$ of $\mathfrak{g}-$representations. Then,

$$\mathfrak{h}\varphi(v^\lambda) = \varphi\mathfrak{h}(v^\lambda) = \varphi\Big(\lambda(\mathfrak{h})v^\lambda\Big) = \lambda(\mathfrak{h})\varphi(v^\lambda),$$

and so $\varphi(M^\lambda) \subseteq N^\lambda$.

Proof of Weyl's formula

Given an exact sequence of representations

$$0 \to V_1 \to \cdots \to V_n \to 0,$$

we can thus restrict to the (finite dimensional) weight subspaces and get:

$$\sum_{i=1}^{n} (-1)^i \dim V_i^\lambda = 0.$$

By the definition of characters,

$$\chi_V = \sum_{\lambda \in \mathrm{Wt}(V)} (\dim V^\lambda) e^\lambda,$$

we thus have

$$\sum_{i=1}^{n} (-1)^i \chi_{V_i} = 0.$$

Proof of Weyl's formula

Using the above in conjunction with the BGG resolution:

$$0 \to \cdots \bigoplus_{w \in W_k} M_{w \circ \lambda} \cdots \to M_\lambda \to V \to 0,$$

we get

$$\chi_V = \sum_{k=0}^{|\Phi_+|} (-1)^k \sum_{\ell(w)=k} \chi_{M_{w \circ \lambda}}.$$

Proof of Weyl's formula

Now using the lemma for the characters of Verma modules

$$\chi_{M_\lambda} = e^{\lambda} \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^{-\alpha}},$$

we can finally conclude

$$\chi_V = \sum_{w \in W} \operatorname{sgn}(w) \chi_{M_{w \circ \lambda}} = \sum_{w \in W} \operatorname{sgn}(w) e^{w \circ \lambda} \prod_{\alpha \in \Phi_+} \frac{1}{1 - e^{-\alpha}}$$

which is exactly Weyl's formula. ∎

Notation and Preliminaries

The BGG resolution

The Weyl Character formula

Outline of the Proof

Outline of the Proof

Theorem (BGG Resolution)

For $V \in \mathrm{irRep}_{\mathrm{fd}}(\mathfrak{g})$ of highest weight $\lambda \in \Lambda^+$, there is a resolution

$$0 \longrightarrow C_{|\Phi_+|} \overset{\mathrm{d}_{k-1}}{\dashrightarrow} C_k \overset{\mathrm{d}_k}{\dashrightarrow} C_0 \overset{\mathrm{d}_0}{\longrightarrow} C_{-1} \longrightarrow 0$$

where $C_k = \bigoplus_{w \in W_k} M_{w \circ \lambda}$ if $0 \le k \le |\Phi_+|$, $C_{-1} = V$ and d_k is defined as $\mathrm{d}_k\big|_{M_{w \circ \lambda}} = \left(\mathrm{sgn}\,(w, w')\iota_{w \to w'}\right)_{w' \in W_{k-1}}$, and $\mathrm{d}_0 : M_\lambda \twoheadrightarrow V$ is the canonical projection.

Steps of the proof

(i) The sequence is a chain complex

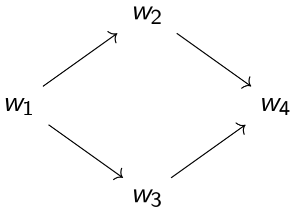
(ii) $\bigoplus M_{s_i \circ \lambda} \longrightarrow M_\lambda \longrightarrow V \longrightarrow 0$ is exact.

(iii) The sequence is exact everywhere else.

First step

The sequence is a chain complex essentially because of the way the differentials d_k are defined using the graph $\Gamma(W)$.

If $w \in W_k, \mathrm{d}_{k-1} \circ \mathrm{d}_k(w) \in W_{k-2}$. But we know that if w_1 and w_4 are 2 apart in length, there are either no morphisms between them or they fit in a square



such that the product of signs is -1. In either case, this means $\mathrm{d}^2 = 0$.

Second step

We want to prove that

$$\bigoplus M_{s_i \circ \lambda} \xrightarrow{\mathrm{d}_1} M_\lambda \xrightarrow{\mathrm{d}_0} V \longrightarrow 0$$

is exact. The map d_0 is surjective as it is the canonical projection.

Lemma

For $\lambda \in \Lambda^+, \alpha_i \in \Sigma$, the submodule of M_λ generated by $f_i^{\lambda(h_i)+1} v^\lambda$ is isomorphic to $M_{s_i \circ \lambda}$. Under these identifications, we get

$$V \simeq M_\lambda \Big/ \sum_{\alpha_i \in \Sigma} M_{s_i \circ \lambda}.$$

Last step

The last step is to show that the sequence is exact everywhere else. This requires a lot of work, and can be reduced to proving the following three lemmas.

Last step

Lemma (BGG 10.5)

If $M, N \in \mathscr{O}$ are such that $M = \text{Span}_{U\mathfrak{n}^-}\{v_1, \ldots, v_n\}$ and $\varphi : M \to N$ a map of $U\mathfrak{n}^-$−modules such that $\varphi(v_i)$ is a weight vector, then φ is surjective if and only the induced map

$$\widetilde{\varphi} : M/\mathfrak{n}^- M \to N/\mathfrak{n}^- N$$

is surjective.

Last step

Lemma (BGG 10.6)

The map

$$\widetilde{\mathrm{d}}_{k+1} : C_{k+1}/\mathfrak{n}^- C_{k+1} \hookrightarrow \ker \mathrm{d}_k/\mathfrak{n}^- \ker \mathrm{d}_k$$

is injective.

Last step

Lemma (BGG 10.7)

$\dim_{\mathbb{C}} C_{k+1}/\mathfrak{n}^- C_{k+1} = \dim_{\mathbb{C}} \ker \mathrm{d}_k/\mathfrak{n}^- \ker \mathrm{d}_k$ is finite.

Further directions

- Kac-Moody algebras
- Functorial BGG

Thank You!

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