Mostow Rigidity

Kleinian Groups Fall 2024

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Introduction

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- Hyperbolic surfaces show the opposite behaviour: loosely, there is a $6g - 6$ dimensional space of *distinct* hyperbolic structures on any closed surface of genus $g \geq 2$.
- Mostow's result is that the rigidity behaviour is what persists in higher dimensions.

Mostow's Theorem

Statement of the Theorem

Theorem 1.1 (Mostow, 1973)

Let $n \geq 3$ and M_1, M_2 be two $n-$dimensional compact connected oriented hyperbolic manifolds. If $f : M_1 \to M_2$ is a homotopy equivalence, there exists an isometry $q : M_1 \to M_2$ that is homotopic to f.

Statement of the Theorem

Theorem 1.2 (Sharper Formulation)

Let $M_i \simeq \mathbb{H}^n/\Gamma_i, i = 1, 2$, be as above. If there is a group isomorphism $\varphi : \Gamma_1 \to \Gamma_2$, then there is an isometry $q \in \mathrm{Isom}(\mathbb{H}^n)$ such that

$$q \circ \gamma = \varphi(\gamma) \circ q,$$

holds for all $\gamma \in \Gamma_1$. In particular, q induces an isometry $\widetilde{\varphi} : M_1 \to M_2$ for which $\widetilde{\varphi}_* = \varphi$.

Motivation

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- Marden's isomorphism theorem:

Theorem (Marden, 1974)

Let G be a geometrically finite Kleinian group without any elliptics, and $\Phi : \Omega(G) \to \Omega(H)$ is a conformal map that induces an isomorphism $\phi : G \to H$ by the correspondence $\phi(g) = \Phi \circ g \circ \Phi^{-1}$. Then Φ is induced by an isometry (Möbius transformation) A and $\phi(g) = AgA^{-1}$.

Extensions

Theorem 1.3 (Prasad, 1974)

Mostow's result holds if we weaken the compactness assumption to the requirement of finite volume.

Extensions

Theorem 1.4 (Thurston et al, 1980s)

If $f : M_1 \to M_2$ is a smooth map such that $\mathrm{vol}(M_1) = |\deg f| \, \mathrm{vol}(M_2)$, then f is homotopic to a locally isometric covering of M_1 onto M_2, of degree $|\deg f|$.

Methods of Proof

Consider the lift $\widetilde{f}$ of f:

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\ \widetilde{f}\ } & \mathbb{H}^n \\ \Big\downarrow p_1 & & \Big\downarrow p_2 \\ M_1 & \xrightarrow[\ f\ ]{} & M_2 \end{array}$$

Note that $\widetilde{f} \circ \gamma = f_*(\gamma) \circ \widetilde{f}$ holds on $\mathbb{H}^n$, for all $\gamma \in \Gamma_1$, for a suitable choice of basepoints.

Methods of Proof

Theorem

Homotoping f to be smooth, $\widetilde{f}$ is a quasi-isometry which extends to a continuous map $\widetilde{f} : \overline{\mathbb{H}}^n \to \overline{\mathbb{H}}^n$, such that $\widetilde{f}\big|_{\partial \mathbb{H}^n}$ is injective and $\widetilde{f} \circ \gamma = f_*(\gamma) \circ \widetilde{f}$ holds on all of $\overline{\mathbb{H}}^n$, for all $\gamma \in \Gamma_1$.

Methods of Proof

- Gromov: $\widetilde{f}\big|_{\partial\mathbb{H}^n}$ is induced by an isometry, which satisfies the requirements. This is shown by looking at images of ideal simplices, and using the Gromov norm.

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Theorem

Let G be a nonelementary Kleinian group, $\zeta \in \Lambda(G)$ a conical limit point, and $f : \mathbb{S}^2 \to \mathbb{S}^2$ a homeomorphism which is differentiable at ζ with nonzero derivative. Suppose $\phi : G \to H$ is a homomorphism to another Kleinian group H such that $f \circ g = \phi(g) \circ f$. Then f is a Möbius transformation.

Methods of Proof

- Gromov: $\widetilde{f}\big|_{\partial\mathbb{H}^n}$ is induced by an isometry, which satisfies the requirements. This is shown by looking at images of ideal simplices, and using the Gromov norm.
- Tukia: Same strategy, but uses analytic techniques from the theory of quasi-conformal maps.
- Besson-Courtois-Gallot: Probabilistic approach, using the so-called volume entropy of Riemannian manifolds.

An Application

$\mathrm{Out}(\pi_1(M))$ of hyperbolic manifolds

Let S_g be a closed oriented surface of genus g.

Theorem (Dehn-Nielsen-Baer)

$\mathrm{Out}(\pi_1(S_g))$ is isomorphic to $\mathrm{Mod}(S_g)$, and in particular is an infinite group.

Theorem (Hurwitz)

$\mathrm{Isom}(S_g)$ has size at most $84(g-1)$.

$\mathrm{Out}(\pi_1(M))$ of hyperbolic manifolds

Theorem

Let M be a closed oriented hyperbolic manifold of dimension $n \geq 3$. Then $\mathrm{Out}(\pi_1(M)) \simeq \mathrm{Isom}(M)$, and is hence a finite group.

The Proof

Let $\Gamma = \pi_1(M)$. We have a map

$$\theta : \mathrm{Isom}(M) \to \mathrm{Out}(\Gamma)$$

given by $f \mapsto [f_*]$, because f_* is an isomorphism of $\pi_1(M, x)$ onto $\pi_1(M, f(x))$.

The Proof

Injectivity

Suppose $\theta(f) = [1]$. There is a lift $\widetilde{f} : \mathbb{H}^n \to \mathbb{H}^n$ of f such that $\widetilde{f} \circ \gamma = \gamma \circ \widetilde{f}$ for all $\gamma \in \Gamma$.

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Let $\delta \neq 1$ be in the centralizer. As $\gamma \in \Gamma \setminus \{1\}$ is hyperbolic, with unique axis ℓ_γ, we get

$$\delta(\ell_\gamma) = \gamma(\delta(\ell_\gamma)) \implies \delta(\ell_\gamma) = \ell_\gamma,$$

and so δ is not parabolic. Let $F = \mathrm{Fix}(\delta)$. Then, for all $\gamma \in \Gamma \setminus \{1\}, \ell_\gamma \subset F$ and $\gamma(F) = F$.

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Fix some $x_0 \in F$ and a line ℓ_0 through x_0 that is orthogonal to F. Then, for small ε,

$$\overline{N_\varepsilon(\ell_0)} \cap (\Gamma \setminus \{1\}) \cdot \overline{N_\varepsilon(\ell_0)} = \emptyset,$$

and so we get a closed subset of M that is not compact. $\to\leftarrow$

The Proof

Surjectivity

Any automorphism of Γ is induced by a homotopy equivalence of M, because M is a $K(\Gamma, 1)$ space. Mostow rigidity gives an isometry f which induces the automorphism, and hence θ is surjective.

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Finiteness

It can be shown that $\text{Isom}(M)$ contains finitely many homotopy classes using the fact that M is compact and hence the sup norm makes $\text{Isom}(M)$ into a compact group. ∎

Gromov's Proof

Outline of Gromov's proof

(i) Extend $\widetilde{f}$ to the boundary $\partial\mathbb{H}^n$ as mentioned before.

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\widetilde{f}} & \mathbb{H}^n \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ M_1 & \xrightarrow[f]{} & M_2 \end{array}$$

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(i) Extend $\widetilde{f}$ to the boundary $\partial\mathbb{H}^n$ as mentioned before.

(ii) Show that the volume function vol() attains its supremum v_n over all geodesic $n-$simplices at the regular and ideal $n-$simplex.

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(iii) If $\{u_0, \ldots, u_n\}$ are the vertices of a simplex of volume v_n, then the simplex on $\left\{\widetilde{f}(u_0), \ldots, \widetilde{f}(u_n)\right\}$ also has volume v_n.

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(iv) Show that the above fact implies that $\widetilde{f}$ is induced by an isometry of $\mathbb{H}^n$.

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(iv) Show that the above fact implies that $\widetilde{f}$ is induced by an isometry of $\mathbb{H}^n$.
Only step where $n \geq 3$ is needed!

First step

Consider the lift $\widetilde{f}$ of f:

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\quad \widetilde{f} \quad} & \mathbb{H}^n \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ M_1 = \mathbb{H}^n/\Gamma_1 & \xrightarrow[\quad f \quad]{} & M_2 = \mathbb{H}^n/\Gamma_2 \end{array}$$

Note that $\widetilde{f} \circ \gamma = f_*(\gamma) \circ \widetilde{f}$ holds on $\mathbb{H}^n$, for all $\gamma \in \Gamma_1$, for a suitable choice of basepoints.

First step

Theorem

Homotoping f to be smooth, $\widetilde{f}$ is a pseudo-isometry which extends to a continuous map $\widetilde{f} : \overline{\mathbb{H}}^n \to \overline{\mathbb{H}}^n$, such that $\widetilde{f}\big|_{\partial \mathbb{H}^n}$ is injective and $\widetilde{f} \circ \gamma = f_*(\gamma) \circ \widetilde{f}$ holds on all of $\overline{\mathbb{H}}^n$, for all $\gamma \in \Gamma_1$.

First step

f is a homotopy equivalence of compact manifolds, and is hence homotopic to a smooth one. By compactness we get f and its homotopy inverse have finite maximum dilatation, and this information can be lifted to get $\widetilde{f} : \mathbb{H}^n \to \mathbb{H}^n$ is a Lipschitz map.

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Finally, using the fact that there is a compact Dirichlet domain for Γ_1 and since lifts commute with the action of Γ_1, we can conclude that $\widetilde{f}$ is a *pseudo-isometry*:

$$\frac{1}{C_1} d(x_1, x_2) - C_2 \leq d(\widetilde{f}(x_1), \widetilde{f}(x_2)) \leq C_1 d(x_1, x_2).$$

First step

Theorem

Any pseudo-isometry $P : \mathbb{H}^n \to \mathbb{H}^n$ extends to a continuous map $P : \overline{\mathbb{H}}^n \to \overline{\mathbb{H}}^n$ that is an injection restricted to the boundary $\partial\mathbb{H}^n$.

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Using the Jordan-Schoenflies theorem, P is in fact a homeomorphism of the sphere at infinity.

Second step

Let $\mathscr{S}_n$ be the set of all ideal $n-$simplices in $\overline{\mathbb{H}}^n$ that have hyperbolic faces.

Definition ▶ Regular simplices

A simplex in $\overline{\mathbb{H}}^n$ is said to be regular if any permutation of its vertices is induced by an isometry.

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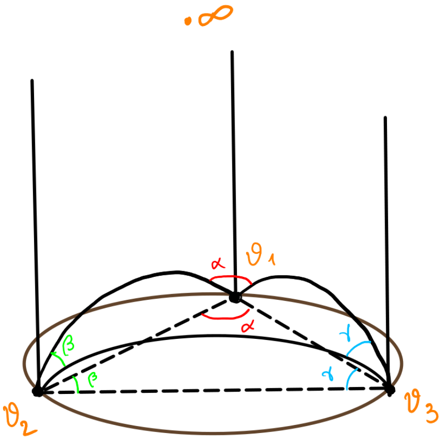
Lemma

Let $\sigma \in \mathscr{S}_n$ have vertices $\infty, v_1, \ldots, v_n$ where $v_i \in \mathbb{R}^n \times \{0\}$. Then σ is regular if and only if the Euclidean simplex on $v_1, \ldots, v_n$ is regular.

Second step



Second step

Theorem

The volume function vol() restricted to $\mathscr{S}_n$ attains its supremum v_n exactly at the regular and ideal $n-$simplices.

Second step

Sketch of the proof:

- For $n = 2$, every ideal triangle is regular and has area $v_2 = \pi$.

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Sketch of the proof:

- For $n = 2$, every ideal triangle is regular and has area $v_2 = \pi$.
- For $n = 3$, it is a direct computation that

$$\mathrm{vol}(\sigma) = \Lambda(\alpha(\sigma)) + \Lambda(\beta(\sigma)) + \Lambda(\gamma(\sigma)),$$

where Λ is the Lobachevsky function

$$\Lambda(\theta) = \int_0^\theta -\log|\sin t|\,\mathrm{d}t,$$

and so σ is of maximal volume iff $\alpha = \beta = \gamma = \frac{\pi}{3}$.

Second step

- For $n \geq 2$, we have the inequality

$$\frac{n-1}{n^2} \leq \frac{v_{n+1}}{v_n} \leq \frac{1}{n}.$$

  Using this inequality and some analysis of the integrals defining the volumes, the result follows.

Third Step

Theorem

Let $\widetilde{f} : \overline{\mathbb{H}}^n \to \overline{\mathbb{H}}^n$ be as in the first step. Then, if $\{u_0, \ldots, u_n\}$ are the vertices of a simplex of volume v_n, the simplex on $\left\{\widetilde{f}(u_0), \ldots, \widetilde{f}(u_n)\right\}$ also has volume v_n.

Gromov norm

Let X be a topological space, and consider $C_k(X;\mathbb{R})$. We make this a normed vector space by setting

$$\|c\| = \inf\left\{\sum_i |a_i| \;\middle|\; c = \sum_i a_i\sigma_i\right\}$$

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$$\|c\| = \inf\left\{\sum_i |a_i| \,\middle|\, c = \sum_i a_i\sigma_i\right\}$$

This norm descends to a semi-norm on the quotient space $H_k(X;\mathbb{R}) = Z_k(X)/B_k(X)$:

$$\|z\| = \inf\{\|c\| \mid c \in Z_k, z = [c]\}$$

Gromov norm

Definition ▶ Gromov norm

For a compact oriented connected manifold M, with fundamental class $[M] \in H_n(M;\mathbb{R})$, we define its Gromov norm as

$$\|M\| = \|[M]\|$$

Gromov norm

Properties:

(i) Let $f : M \to N$ be a continuous map between manifolds. Then,

$$\|M\| \geq |\deg f| \cdot \|N\|.$$

In particular, $\|\cdot\|$ is homotopy invariant.

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Proof.

If $\alpha \in H_k(M)$, $\|f_*(\alpha)\| \leq \|\alpha\|$ and degree satisfies

$$f_*([M]) = \deg f \cdot [N]$$

□

Gromov norm

Properties:

(i) Let $f : M \to N$ be a continuous map between manifolds. Then,

$$\|M\| \geq |\deg f| \cdot \|N\|.$$

In particular, $\|\cdot\|$ is homotopy invariant.

(ii) If M admits a continuous self-map of degree at least 2, then $\|M\| = 0$. Hence, all spheres and the torus have Gromov norm 0.

Gromov's theorem

Theorem (Gromov)

If M is a compact oriented connected hyperbolic manifold of dimension n,

$$\mathrm{vol}(M) = v_n \|M\|.$$

In particular, hyperbolic volume is a homotopy invariant.

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Corollary

Any such manifold M has non-zero Gromov norm and if $f : M \to M$ is continuous, $|\deg f| \leq 1$.

Proof of Gromov's Theorem

- The proof involves defining *straight* $n-$chains and their algebraic volume. The inequality $\text{vol}(M) \leq v_n \|M\|$ is straightforward.

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- The proof of $\text{vol}(M) \geq v_n \|M\|$ is quite technical, and requires the notion of $\varepsilon-$efficient cycles and computations involving the Haar measure on $\text{Isom}(\mathbb{H}^n) \simeq SO(n,1)$.

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- The proof involves defining *straight* $n-$chains and their algebraic volume. The inequality $\text{vol}(M) \leq v_n \|M\|$ is straightforward.
- The proof of $\text{vol}(M) \geq v_n \|M\|$ is quite technical, and requires the notion of $\varepsilon-$efficient cycles and computations involving the Haar measure on $\text{Isom}(\mathbb{H}^n) \simeq SO(n, 1)$.
- There is a more conceptual proof due to Milnor and Thurston, but that involves notions of measure homology. However, these methods can be used to generalize the theorem to $(G, X)-$manifolds and equivariant cohomology.

Third step

Theorem

Let $\widetilde{f} : \overline{\mathbb{H}}^n \to \overline{\mathbb{H}}^n$ be as in the first step. Then, if $\{u_0, \ldots, u_n\}$ are the vertices of a simplex of volume v_n, the simplex on $\left\{\widetilde{f}(u_0), \ldots, \widetilde{f}(u_n)\right\}$ also has volume v_n.

Third step

The proof proceeds by contradiction, assuming that the simplex $\sigma(\widetilde{f}(w_0), \ldots, \widetilde{f}(w_n))$ has volume $v_n - 2\varepsilon$, where $\sigma(w_0, \ldots, w_n)$ has volume v_n. By continuity, there are neighborhoods U_j of w_j such that $\text{vol}(\sigma(\widetilde{f}(u_0), \ldots, \widetilde{f}(u_n))) \leq v_n - \varepsilon$. Using the techniques as in the proof that $\text{vol}(M) \geq v_n \|M\|$, we get a contradiction.

Fourth step

Theorem

Let $n \geq 3$ and $P : \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ be a continuous injection, such that $\mathrm{vol}(\sigma(P(u_0), \ldots, P(u_n))) = v_n$ whenever $\mathrm{vol}(\sigma(u_0, \ldots, u_n)) = v_n$. Then P is induced by an isometry.

Fourth step

Using the result of the second step, P maps vertices of any regular ideal $n-$simplex to the vertices of another. As any two such simplices are isometric, we can compose P with an isometry $Q \in \mathrm{Isom}(\mathbb{H}^n)$ so that now $P \circ Q$ fixes some simplex with vertices $\infty, v_1, \ldots, v_n$, where the v_j lie on $\mathbb{R}^n \times 0$.

Lemma

Let $\sigma \in \mathscr{S}_n$ have vertices $\infty, v_1, \ldots, v_n$ where $v_i \in \mathbb{R}^n \times \{0\}$. Then σ is regular if and only if the Euclidean simplex on $v_1, \ldots, v_n$ is regular.

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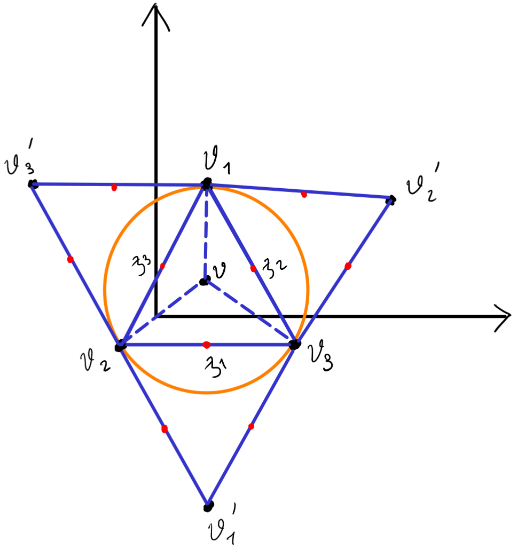

Fourth step

Hence, $P \circ Q$ fixes a dense set of points in $\partial\mathbb{H}^n$ and therefore must be the identity by continuity. ∎

End of Gromov's proof

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$$\widetilde{f} \circ \gamma = f_*(\gamma) \circ \widetilde{f}, \gamma \in \Gamma_1$$

holds over all $\overline{\mathbb{H}}^n$.

End of Gromov's proof

Combining the results so far, the lift $\widetilde{f}$ of f extends to a continuous injection of the boundary $\partial\mathbb{H}^n$, and is induced by some isometry Q there. We thus get the relation

$$Q \circ \gamma = f_*(\gamma) \circ Q,$$

for all $\gamma \in \Gamma_1$, on $\partial\mathbb{H}^n$. As every term involves an isometry of $\mathbb{H}^n$, the relation must hold on all of $\mathbb{H}^n$.

End of Gromov's proof

Consider the map $q : M_1 \to M_2$ defined as

$$q(p_1(x)) = p_2(Q(x)),\ x \in \mathbb{H}^n$$

Then it is easy to check that q is a well-defined bijection, and is an isometry because p_1, p_2 and Q are local isometries. Finally,

$$H(t, p_1(x)) = p_2(t\widetilde{f}(x) + (1-t)Q(x))$$

defines a homotopy of f with q. ■

Thank You!

References

- Riccardo Benedetti and Carlo Petronio.
  Lectures on Hyperbolic geometry.
  Springer, 1992.
- Albert Marden.
  Outer Circles.
  Cambridge, 2007.
- Bruno Martelli.
  An introduction to geometric topology, 2022.
- William Thurston.
  The geometry and topology of 3-manifolds.
  Lecture notes, 1978.