What is ?(x)?

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9th March, 2024

Can we **count** rationals?

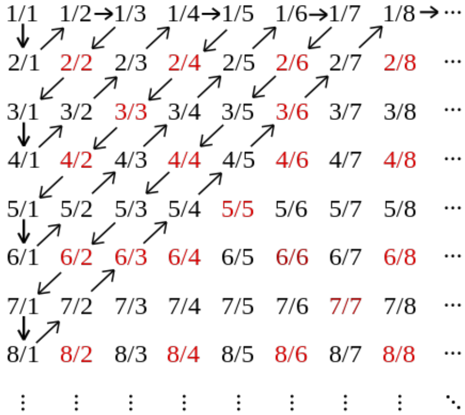


Figure: Wikipedia, Rational Numbers – Countability

Can we **count** rationals?

Two more 'explicit' bijections (from Prof. Bhat's 2022 Real Analysis 1 course):

(i) $g(m, n) = 2^{m-1}(2n - 1)$

(ii) $h(m, n) = m + \frac{(m+n-1)(m+n-2)}{2}$

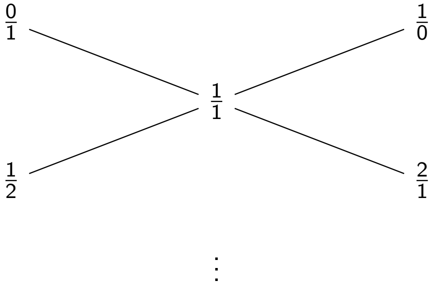
Can we **count** rationals?

The mediant:

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$$

Can we **count** rationals?

The Stern-Brocot Tree



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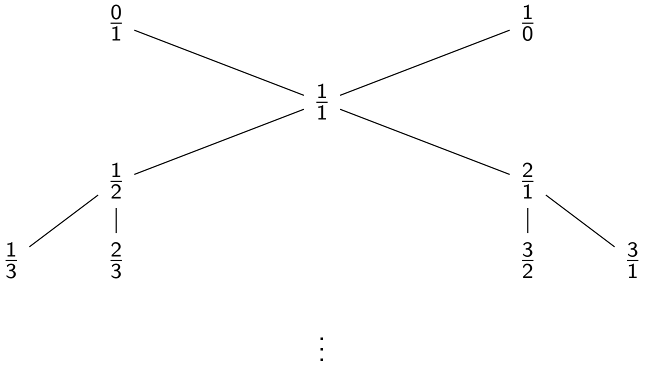

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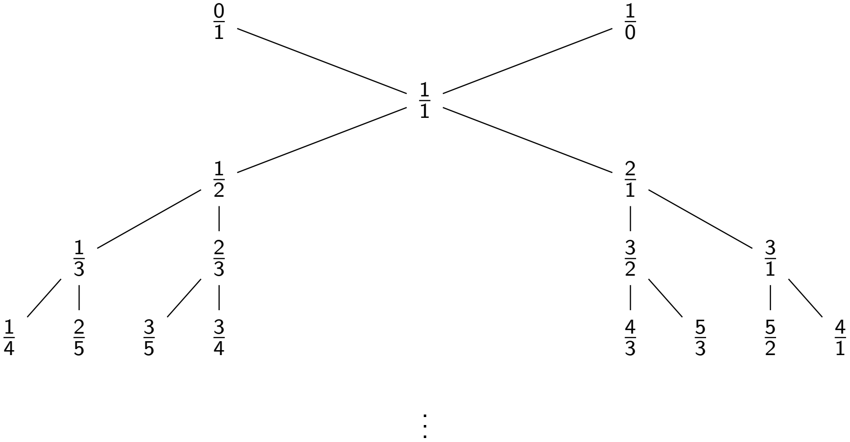
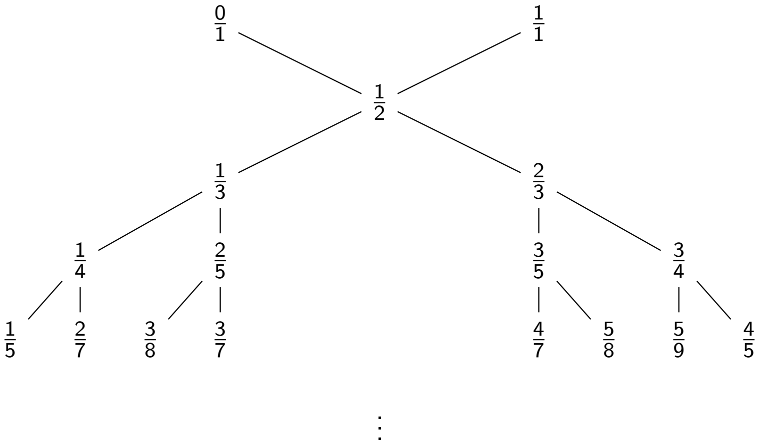


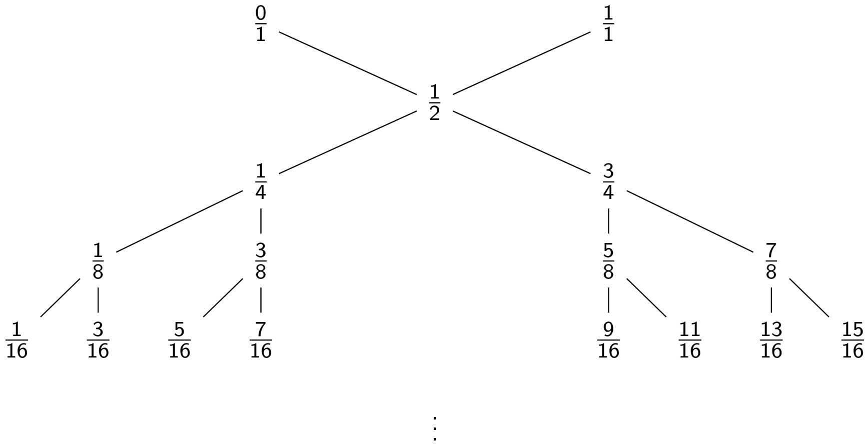
The Farey tree and $?(x)$

The Farey Tree



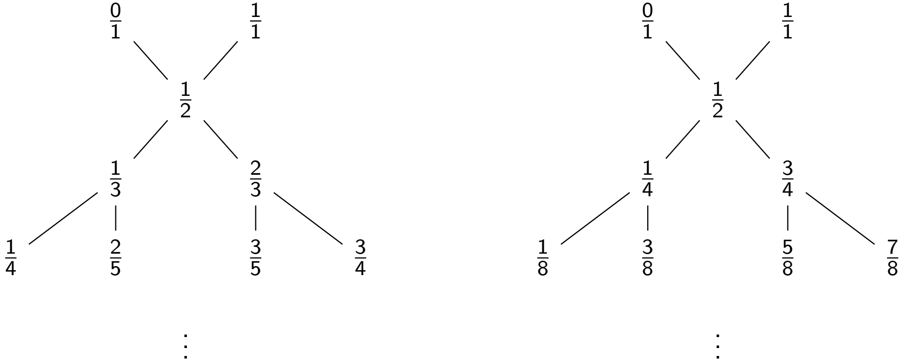
The Farey tree and $?(x)$

The Dyadic Tree



The Farey tree and $?(x)$

The ? map



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- $?(0) = 0, ?(1) = 1, ?\left(\frac{1}{2}\right) = \frac{1}{2}$
- $?\left(\frac{1}{3}\right) = \frac{1}{4}$
- $?\left(\frac{p}{q} \oplus \frac{r}{s}\right) = \frac{?(p/q)+?(r/s)}{2}$
- $?$ is a strictly increasing function

An analytic definition

Let $x = [a_1, a_2, \ldots, a_n]$ with convergents

$$\frac{p_j}{q_j} = \frac{1}{a_1 + \frac{1}{\cdots + \frac{1}{a_j}}},$$

and let $y_j = ?\left(\frac{p_j}{q_j}\right)$.

An analytic definition

$$?\left(\frac{p_k}{q_k} \oplus \frac{p_{k-1}}{q_{k-1}}\right) = \frac{y_k + y_{k-1}}{2}$$

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$$\implies y_{k+1} = ?\left(\frac{a_{k+1}p_k + p_{k-1}}{a_{k+1}q_k + q_{k-1}}\right) = \frac{y_k}{2} + \cdots + \frac{y_k}{2^{a_{k+1}}} + \frac{y_{k-1}}{2^{a_{k+1}}}$$

An analytic definition

Hence, for $x = [a_1, \ldots, a_n]$

$$?(x) = y_n = 2\sum_{k=1}^{n} \frac{(-1)^{k-1}}{2^{a_1+\cdots+a_k}}.$$

We extend this definition by continuity: for $x = [a_1, \ldots, a_n, \ldots] \in [0, 1]$, we get

$$?(x) = 2\sum_{k\geq 1} \frac{(-1)^{k-1}}{2^{a_1+\cdots+a_k}}$$

An analytic definition

So far we have:

- ? is a strictly increasing continuous function on $[0, 1]$.
- ? maps rationals in $[0, 1]$ to dyadic rationals.
- For irrational $x \in [0, 1]$, $?(x)$ has infinite dyadic expansion.
- $x \in [0, 1]$ is a quadratic irrational iff $?(x)$ is rational with infinite dyadic expansion.

Self symmetry of ?

Consider the graph $\Gamma = \{(x, ?(x)) \mid x \in [0,1]\} \subseteq [0,1]^2$.

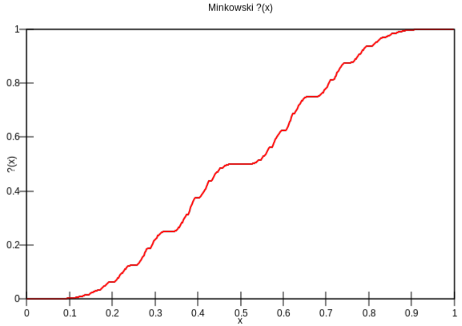

Figure: Wikipedia, Minkowski ? function

Self symmetry of ?

The *obvious* symmetry of ? is $?(1-x) = 1-?(x)$. From the definition of ?, we also get the symmetry $?\left(\frac{x}{x+1}\right) = \frac{?(x)}{2}$, because $x \mapsto \frac{x}{x+1}$ is the same as

$$[a_1, \ldots, a_n, \ldots] \mapsto [a_1+1, a_2, \ldots, a_n, \ldots].$$

Self symmetry of ?

Let $S(x,y) = (1-x, 1-y)$ and $R(x,y) = \left(\frac{x}{x+1}, \frac{y}{2}\right)$. It can be shown that **all** (self-similarity) symmetries of Γ are of the form

$$R^{a_1} S R^{a_2} \cdots S R^{a_m},\ a_1, \ldots, a_m \in \mathbb{Z}_{\geq 0}.$$

Self symmetry of ?

We can represent S and R as Möbius transformations as follows:

$$S(x) = 1 - x = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot x$$

$$R(x) = \frac{x}{x+1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot x$$

The monoid $\langle R, S \rangle \subseteq \mathrm{PGL}(2, \mathbb{Z})$ is called the *dyadic monoid* or the *period-doubling monoid*.

de Rham curves

Let (M, d) be a complete metric space and let $f_0, f_1 : M \to M$ be two contracting maps. The Banach fixed-point theorem guarantees that there are fixed points p_0 and p_1 of f_0 and f_1 respectively.

Now fix $x = \sum_{k \geq 0} \frac{b_k}{2^k} \in [0, 1]$ and consider the map $c_x : M \to M$ defined as:

$$c_x(p) = f_{b_0} \circ f_{b_1} \cdots f_{b_k} \circ \cdots (p)$$

de Rham curves

Continuity of c_x: $f_0(p_1) = f_1(p_0)$

Assuming that, it can be shown that c_x maps the common basin of attraction of f_0, f_1 to a single point p_x. The curve $x \mapsto p_x$ is the de Rham curve associated to f_0, f_1.

de Rham curves

The Minkowski ? function's graph Γ is a de Rham curve with $f_0(z) = \frac{z}{z+1}$ and $f_1(z) = \frac{1}{2-z}$.

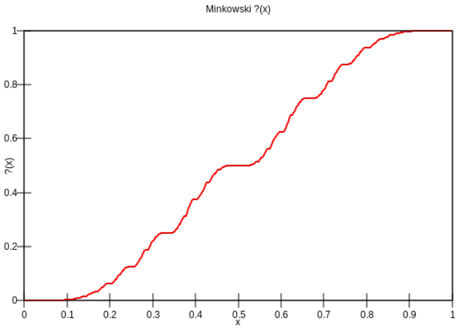

Figure: Wikipedia, Minkowski ? function

de Rham curves

The Cesáro - Faber are generated by $f_0(z) = az$ and $f_1(z) = a + (1-a)z$.

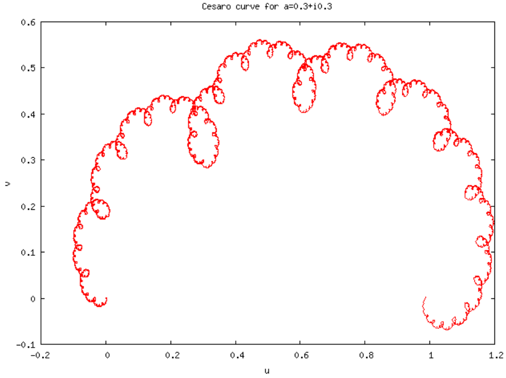

Figure: Wikipedia, de Rham curves ($a = 0.3 + \iota 0.3$)

de Rham curves

The Lévy C curve as a Cesáro curve

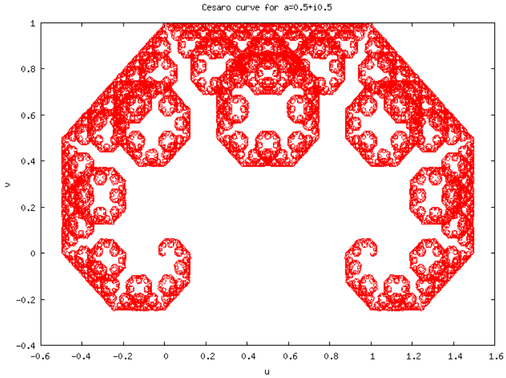

Figure: Wikipedia, de Rham curves ($a = 0.5 + \iota 0.5$)

de Rham curves

The first four iterations of the Koch snowflake

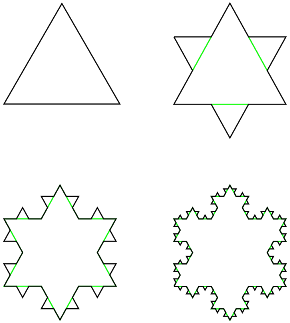

Figure: Wikipedia, Koch snowflake

de Rham curves

The first three iterations of the Peano curve

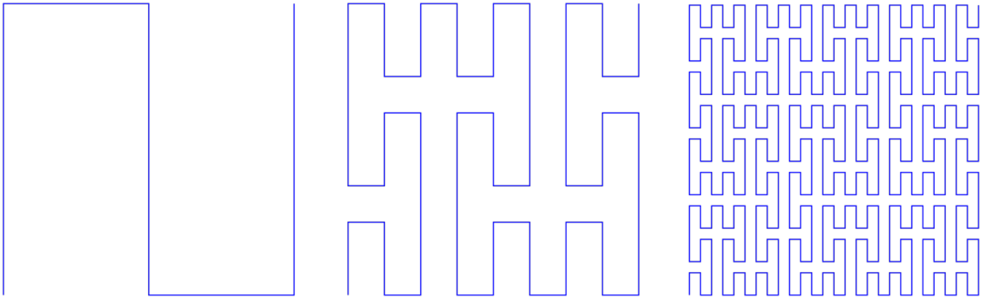

Figure: Wikipedia, Peano curve

de Rham curves

The Koch-Peano curve is a de Rham curve with $f_0(z) = a\bar{z}$ and $f_1(z) = a + (1-a)\bar{z}$.

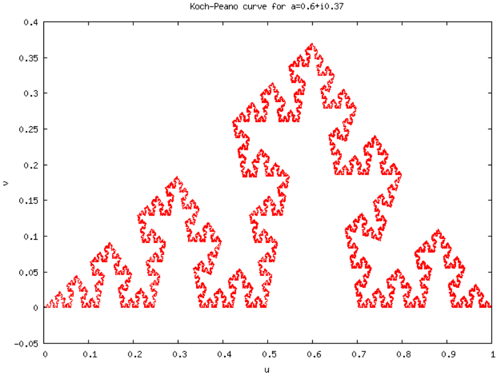

Figure: Wikipedia, de Rham curves ($a = 0.6 + \iota 0.37$)

de Rham curves

The Takagi-Landsberg or Blancmange curve is an affine de Rham curve. Here,

$$f_1(x,y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & w \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

$$f_2(x,y) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & w \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

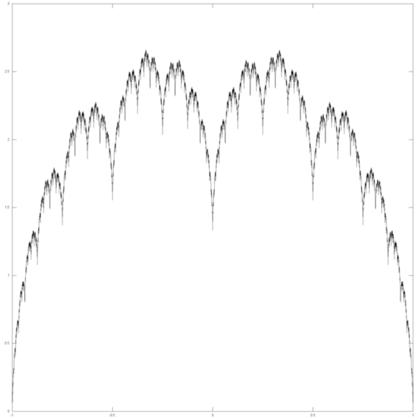

Figure: Wikipedia, Blancmange curve ($w = \frac{2}{3}$)

? is Lipschitz, yet singular

Theorem (Salem, 1943)

$?(x)$ is a Lipschitz function of order $\frac{1}{2}\frac{\log 2}{\log \varphi}$, where $\varphi = \frac{1+\sqrt{5}}{2}$, and in fact, this is its Hölder exponent.

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Theorem (Salem, 1943)

?(x) is a Lipschitz function of order $\frac{1}{2}\frac{\log 2}{\log \varphi}$, where $\varphi = \frac{1+\sqrt{5}}{2}$, and in fact, this is its Hölder exponent.

The proof follows from the purely number theoretic fact that if $\frac{p}{q} = [a_1, \ldots, a_n] \in [0, 1]$, then $q < \varphi^{a_1+\cdots+a_n}$.

? is Lipschitz, yet singular

What about the derivative of $?(x)$? Salem also showed that ? is a *singular* function, i.e, $?'(x) = 0$ for almost all $x \in [0, 1]$. He demonstrated that for $?'$ vanishes on the set

$$\{x = [a_1, \dots] \mid \limsup a_n = \infty\} \cap \{x \mid ?'(x) < \infty\}.$$

Both the sets above are of measure 1, and hence, so is their intersection.

Another singular set of ?

Definition ▶ Asymptotic Distribution Function

Given a sequence $(a_n)_{\mathbb{N}} \subseteq [0,1]$, we define its *asymptotic distribution function* to be $F : [0,1] \to \mathbb{R}$

$$F(x) = \lim_{n\to\infty} \frac{|\{i \in [n] \mid a_i \leq x\}|}{n}.$$

Another singular set of ?

Consider the map $h : (0,1] \to [0,1]$ defined as

$$h(x) = \frac{1}{x} \mod 1 = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$

Theorem (Gauss-Kuzmin)

The set $G = \{x \in (0,1] \mid \text{the orbit } (h^n(x))_{n \geq 0} \text{ has } \log_2(1+t) \text{ as its adf}\}$ is of measure 1.

Another singular set of ?

From the formula $?(x) = 2\sum_{k\geq 1} \frac{(-1)^{k-1}}{2^{a_1+\cdots+a_k}}$, we know the Minkowski function represents reals in the *alternating dyadic* system.

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Let $F(x) = 2(1 - 2^{n_x}x)$, $n_x = \left\lfloor \log_2 \frac{1}{x} \right\rfloor$. We call a number $x \in [0,1]$ *normal* in the alternating dyadic system if $(F^n(x))_{n\geq 0}$ is uniformly distributed in $[0,1]$.

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It can be shown that F preserves the Lebesgue measure in $[0,1]$, and is in fact ergodic. Hence, for almost all x, the orbit under F is indeed uniformly distributed in $[0,1]$, i.e, the set N of normal numbers is of measure 1.

Another singular set of ?

Theorem

$$\mu(?(G \cap N)) = \mu(?^{-1}(G \cap N)) = 0$$

Intersecting $G \cap N$ with the set where $?'$ exists, we get another singular set for ?.

What Next?

- Fourier series

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- Other integral transforms

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- Fourier series
- Other integral transforms
- Modular forms

Thank You!