Ramified Coverings, Riemann-Hurwitz Formula and Orbifolds

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1 Ramified Coverings

We fix the following notation: for a holomorphic map $f : X \to Y$ and charts φ, ψ around $p \in X$ and $f(p) \in Y$ resp., we denote $\psi \circ f \circ \varphi^{-1}$ by $\widehat{f}$.

The following result describes the local behaviour of analytic maps between Riemann surfaces.

Theorem 1 (Local Normal Form)

Let $F : X \to Y$ be a non-constant holomorphic map of Riemann surfaces, and $p \in X$. Then there is a unique $m \in \mathbb{Z}_{\geq 1}$ such that: for all charts $\varphi_2 : U_2 \to V_2$ centered at $F(p)$, there is a chart $\varphi_1 : U_1 \to V_1$ centered at p such that $\widehat{F}(z) = z^m$. This integer is called the *multiplicity* of F at p, denoted by $\mathrm{mult}_p(F)$.

Proof. Fix a chart φ_2 centered at $F(p)$, and any chart ψ centered at p. Then, $\widehat{F}$, with respect to these charts, has a Taylor expansion

$$\widehat{F}_1(z) = \sum_{i=m}^{\infty} c_i z^i = z^m f(z),$$

where $m \geq 1$ as $\widehat{F}_1(0) = 0$, $c_m \neq 0$, and $f(0) \neq 0$. Let g be a holomorphic function around 0 such that $g(z)^m = f(z)$, so that $\widehat{F}_1(z) = (zg(z))^m = \eta(z)^m$. As $\eta'(0) \neq 0$, we get η is invertible near 0. We consider the new coordinates $w = \eta(z)$ near p, this is the chart φ_1. Then, with respect to φ_1, φ_2,

$$\widehat{F}(w) = \widehat{F}_1(\eta^{-1}(w)) = w^m$$

and this shows the existence.

As for uniqueness, if $\widehat{F}(z) = z^m$ for some local coordinates around p and $F(p)$, there we must have exactly m preimages of points near $F(p)$. This is a purely topological property of F, and thus m is independent of the local coordinates chosen. □

Example: If $\varphi : U \to V$ is a chart on X, $\mathrm{mult}_p(\varphi) = 1$ for all $p \in U$.

Note that for a holomorphic function F, $\mathrm{mult}_p(F) \geq 1$ for all p. We now give an easier way of computing multiplicity.

Choose local coordinates z around p and w around $F(p)$, and suppose p is z_0 and $F(p)$ is w_0. In these coordinates, F is given by $h(z) = w$. We then have the following result.

Lemma 1

If F, z, w, h as above, then

$$\mathrm{mult}_p(F) = 1 + \mathrm{ord}_{z_0}(h').$$

In particular, the multiplicity is the index of the least strictly positive term in the power series of h.

The proof follows from the proof of the Theorem.

As a corollary of the lemma, we get that the points of multiplicity more than 1 form a discrete subset of the domain. We define $p \in X$ to be a *ramification point* of F if $\mathrm{mult}_p(F) > 1$, and $F(p)$ to be a *branch point*.

Example 1 (Smooth plane curves)

Let X be a smooth affine plane curve defined by $f(x, y) = 0$, and let $\pi : X \to C$ be the projection $(x, y) \mapsto x$. We will show that π is ramified at $p = (x_0, y_0)$ iff $\partial_y f(p) = 0$.

Suppose $\partial_y f(p) \neq 0$. Then π is a chart map near p, and so has multiplicity 1.

Now suppose $\partial_y f(p) = 0$. By smoothness at p, $\partial_x f(p) \neq 0$ and so, $(x, y) \mapsto y$ is a chart map near p. By the implicit function theorem, there is neighbourhood around p where X consists only of points $(g(y), y)$ for some holomorphic function g, i.e, $f(g(y), y) = 0$ for points y near y_0. Note that $\pi(x, y) = g(y)$ for such y. But we also have,

$$\frac{\partial f}{\partial x} g'(y) + \frac{\partial f}{\partial y} = 0$$

in this neighbourhood, and so $g'(y_0) = 0$. Hence, by the lemma, we get π is ramified at p.

Lemma 2

There is a correspondence

$$\left\{ \begin{array}{c} \text{Meromorphic functions } f \\ \text{on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Holomorphic maps } F : X \to \widehat{\mathbb{C}} \\ \text{that are not identically } \infty \end{array} \right\}$$

As a consequence of lemma 1 and 2, we get the following correspondence.

Lemma 3

Let f be a meromorphic function on X and F be the corresponding holomorphic map $X \to \widehat{\mathbb{C}}$. Then,

(i) If $f(p) = 0$, then $\operatorname{mult}_p(F) = \operatorname{ord}_p(f)$.

(ii) If p is a pole of f, then $\operatorname{mult}_p(F) = -\operatorname{ord}_p(f)$.

(iii) If p is neither a pole nor a zero of f, then $\operatorname{mult}_p(F) = \operatorname{ord}_p(f - f(p))$.

Henceforth we restrict our attention to compact Riemann surfaces, and all our spaces will be compact unless stated otherwise.

1.1 Degree

Theorem 2

Let $F : X \to Y$ be a non-constant holomorphic map. For each $y \in Y$, we define

$$d_y(F) = \sum_{p \in F^{-1}(y)} \operatorname{mult}_p(F).$$

$d_y(F)$ is a constant, independent of y.

Proof. We will show that the map $y \mapsto d_y(F)$ is locally constant as a map $Y \to \mathbb{Z}$, and hence must be constant as Y is connected.

First consider the map $f : \mathbb{D} \to \mathbb{D}$ given by $z \mapsto z^m$, for some $m \in \mathbb{Z}_{\geq 1}$. Then, f is a holomorphic surjection and is ramified only at 0, with multiplicity m. That is, $d_0(f) = m$. Further, for any $w \neq 0$, there are exactly m preimages under f and each has multiplicity m, and so we get $d_w(f) = m$ for $w \neq 0$ as well. Therefore, in the local normal form, the constancy claimed is true.

Fix $y \in Y$, and let $F^{-1}(y) = \{x_1, \ldots, x_n\}$. By the local normal form, we get there are neighbourhoods U_i around x_i, and a neighbourhood V around y such that F is given locally by $z_i \mapsto z_i^{m_i}$. Hence, restricted to the disjoint union of these U_i's we get F is a disjoint union of the disk maps we described above. If we show that, shrinking U_i's and V further if necessary, there are no "unaccounted preimages", we'll be done.

More precisely, suppose there are points arbitrarily close to y such that their preimages do not lie in any of the U_i's, i.e, are not near x_i's. But then, by compactness, there exists a convergent subsequence (p_n) of the sequence of these preimages. We get,

$$F(\lim p_n) = \lim F(p_n) = y.$$

Hence, $\lim p_n \in F^{-1}(y)$, and so must lie in some U_i, a contradiction! □

We define the *degree* of $F : X \to Y$ to be the constant in the previous theorem,

$$\deg F = d_y(F) = \sum_{p \in F^{-1}(y)} \text{mult}_p(F),$$

for any $y \in Y$.

The theorem can be interpreted as the fact that the number of preimages counted with multiplicity is constant. We show an application of this below.

Example 2

Clearly, $F : X \to Y$ is injective if it has degree 1. Hence, F is an isomorphism iff it has degree 1.

Now consider $f : X \to \mathbb{C}$ which is holomorphic except at p, where it has a simple pole. The corresponding $F : X \to \widehat{\mathbb{C}}$ therefore has multiplicity 1 everywhere, and so is of degree 1. Hence, $X \simeq \widehat{\mathbb{C}}$. Therefore, if X is a compact Riemann surface with a meromorphic function that has a single simple pole, $X \simeq \widehat{\mathbb{C}}$.

Suppose $F : X \to Y$ is a non-constant holomorphic map. Deleting the branch points in Y, and the ramification points in X, we get a covering map F between 2−manifolds. This is the reason why F is termed a *ramified* or *branched* covering.

2 The Riemann-Hurwitz formula

We first need some topological preliminaries. Recall that a *triangulation* of a topological surface consists of a decomposition of S into closed subsets that are homeomorphic to triangles, and are pairwise either disjoint, meet only at a single vertex or meet along a single edge. Given a triangulation of S, we denote the number of vertices by V, edges by E and faces by F. We define the *Euler characteristic* of S to be $\chi(S) = V - E + F$.

Also recall the classification of closed orientable surfaces: any such surface is homeomorphic to a connected sum of g many tori, where g is the *genus* of S. It is a fact that the Euler characteristic is independent of the triangulation, and is actually given by the formula $\chi(S) = 2 - 2g$. Finally, as any compact Riemann surface is topologically a compact orientable surface without boundary, we can consider the Euler characteristic of such objects. The following result portrays how χ changes under a ramified covering.

Theorem 3 (Riemann-Hurwitz formula)

Let $F : X \to Y$ be a non-constant holomorphic map. Then,

$$-\chi(X) = -\chi(Y) \deg F + \sum_{p \in X} (\text{mult}_p(F) - 1)$$

Proof. Fix a triangulation of Y in which all branch points of F occur as vertices, say with V vertices, E edges and F faces. Consider the triangulation of X obtained by lifting the one on Y, under the "covering" F, say with V' vertices, E' edges and F' faces. Note that all ramification points must be vertices here. Hence, each face in Y lifts to $\deg F$ faces in X, i.e, $F' = F \cdot \deg F$. Similarly, $E' = E \cdot \deg E$.

Now consider a vertex $q \in Y$. The number of preimages of q is,

$$\left|F^{-1}(q)\right| = \sum_{p \in F^{-1}(q)} 1 = \deg F + \sum_{p \in F^{-1}(q)} (1 - \text{mult}_p(F)).$$

Hence,

$$V' = V \cdot \deg F - \sum_q \sum_{p \in F^{-1}(q)} (\text{mult}_p(F) - 1) = V \cdot \deg F - \sum_{p \in X} (\text{mult}_p(F) - 1)$$

This now gives,

$$\begin{aligned} -\chi(X) &= -V' + E' - F' \\ &= -\deg F \cdot (V - E + F) + \sum_{p \in X} (\text{mult}_p(F) - 1) \\ \implies -\chi(X) &= -\deg F \cdot \chi(Y) + \sum_{p \in X} (\text{mult}_p(F) - 1) \end{aligned}$$

which is what we wanted. □

3 Orbifolds and Hurwitz's automorphism theorem

3.1 Some preliminaries

- Any compact Riemann surface X of genus $g \geq 2$ is isometric to a (closed) hyperbolic surface S_g, and $\text{Aut}(X) \simeq \text{Isom}^+(S_g)$. This is a consequence of the uniformization theorem.
- It can be shown using some basic facts about hyperbolic surfaces that $\text{Isom}^+(S_g)$ is finite, for all $g \geq 2$.

Now suppose that you have a compact Riemann surface X, and consider the quotient space $Y = X/\text{Isom}^+(X)$, and *assume this is a manifold.* Because of the finiteness of the group, we get that Y has finite volume,

$$\text{Area}(Y) = \frac{\text{Area}(X)}{\left|\text{Isom}^+(X)\right|} = \frac{2\pi(2g-2)}{\left|\text{Isom}^+(X)\right|}$$

Hence, if we get a lower bound on the area of these quotients, we'd get a upper bound on the size of the groups. However, note that $\text{Isom}^+(X)$ need not act freely on X, and so a priori we cannot treat Y as a manifold! To study such quotients, we introduce the notion of orbifolds.

3.2 Orbifolds

We would like to consider the orbit space $X/\text{Isom}^+(X)$. However, this object is a manifold only when $\text{Isom}^+(X)$ acts freely on X which might not be the case. To deal with this, we introduce the notion of *orbifolds* that are locally modelled on $\mathbb{R}^n/\Gamma$, for some finite group Γ. We give below only the definitions needed for the purpose of proving the following results, but orbifolds can be dealt with in much more generality.

Definition 1 ▶ Orbifolds

A (2−dimensional hyperbolic) orbifold is a quotient X/G where X is an orientable surface with a hyperbolic metric and G is a subgroup of $\text{Isom}^+(X)$. We **define** the *orbifold fundamental group* of an orbifold Y to be the deck transformation group of the universal cover $\widetilde{Y} = \mathfrak{h}$. Any point of $y \in Y$ has a neighbourhood isometric to a quotient of $\mathfrak{h}$ by a finite group $F_y < \text{Isom}^+(\mathfrak{h})$. y is said to be a *cone point* if F_y is non-trivial, and a *regular point* otherwise. A cone point y is said to have order $|F_y|$.

It can be shown that there are only finitely many cone points in an orbifold X and hence the underlying topological space is always homeomorphic to some S_g (obtained by "forgetting" the cone point data). If the underlying surface of X has genus g and cone points with orders $p_1, \ldots, p_m$, we define the *signature* of X to be $(g; p_1, \ldots, p_m)$.

We can also think of the orbifold with signature $(g; p_1, \ldots, p_m)$ as follows: Start with S_g, and remove m open disks from it. Add in "fractions of disks", essentially disks that have been wrapped up on themselves to get total angles $\frac{2\pi}{p_i}$.

We **define** an isometry of an orbifold X to be an isometry of the metric space X: hence, these are isometries of the Riemannian manifold $X \setminus \{\text{cone points}\}$, and must map any cone point to another one of the same order. Let $\mathrm{Isom}_O(X)$ be the group formed by the orbifold isometries of X. A map $f : X \to Y$ between orbifolds is a *d-fold orbifold covering* if it is the quotient map by a group $G \leq \mathrm{Isom}_O(X)$ of order d. Note that if $y \in Y$ is a cone point of order p that has preimages of orders $q_1, \ldots, q_k$, then

$$\sum_{i=1}^{k} \frac{p}{q_i} = d.$$

This follows from the fact that f must be an unramified covering outside of the cone points in Y. If X has signature $(h; q_1, \ldots, q_n)$ and Y has signature $(g; p_1, \ldots, p_m)$ with a $d-$fold covering $X \to Y$, then we thus get

$$\sum_{i=1}^{n} \frac{1}{q_i} = d \sum_{i=1}^{m} \frac{1}{p_i}.$$

We now define the *orbifold Euler characteristic* of an orbifold X with signature $(g; p_1, \ldots, p_m)$ to be

$$\chi_O(X) = 2 - 2g - m + \sum_{i=1}^{m} \frac{1}{p_i}.$$

The Riemann-Hurwitz formula exactly expresses the multiplicativity of the orbifold Euler characteristic; if there is a $d-$fold covering $X \to Y$, then $\chi_O(X) = d\chi_O(Y)$.

Note that $\chi_O(X) = \chi(X)$ if X has no cone points. Further, if $X = S/G$, $\chi(S) = |G|\chi_O(X) < 0$.

Finally, if $Y = X/G$ is an orbifold which is a quotient of a hyperbolic surface X, we define $\mathrm{Area}(Y) = \mathrm{Area}(X)/|G|$. By the multiplicativity of the Euler characteristic above, the Gauss-Bonnet theorem extends to orbifolds; if Y is any orbifold, $\mathrm{Area}(Y) = -2\pi\chi_O(Y)$. The following theorem will give us the classical $84(g-1)$ result of Hurwitz.

Theorem 4

If X is a compact orbifold, then $\chi_O(X) \leq \frac{-1}{42}$, or equivalently, $\mathrm{Area}(X) \geq \frac{\pi}{21}$. Further, the orbifold with signature $(0; 2, 3, 7)$ is the unique orbifold attaining these bounds.

Proof. The proof is by casework which removes all possibilities but the $(0; 2, 3, 7)$ orbifold. Note that any cone point contributes a $\frac{-1}{2}$ to the Euler characteristic.

Suppose X is a compact orbifold with $\chi_O(X) \geq \frac{-1}{42}$. If X has no cone points, the Euler characteristic is an integer, and if it has genus more than 1, the Euler characteristic is less than -2. Hence, X must have genus 0 or 1, and must have at least one cone point. But if indeed the genus is 1 and there is at least one cone point, $\chi_O(X) \leq \frac{-1}{2}$. So the genus of X must be 0. Therefore, X must have at least 3 cone points as otherwise $\chi_O(X) \geq 0$.

If X has more than 4 cone points, $\chi_O(X) \leq 2 - \frac{5}{2} = \frac{-1}{2}$.

Suppose X has 4 cone points. If the signature is $(0; 2, 2, 2, 2)$, then $\chi_O(X) = 0$, a contradiction. If one cone points has order more than 2,

$$\chi_O(X) \leq 2 - 3 \cdot \frac{1}{2} - \frac{2}{3} = \frac{-1}{6} < \frac{-1}{42}.$$

So X has exactly 3 cone points. Some more casework shows that X cannot have any signature other than $(0; 2, 3, 7)$, and in that case $\chi_O(X) = \frac{-1}{42}$. □

An orbifold of signature $(0; 2, 3, 7)$ can be constructed using triangle groups.

Theorem 5

Let X be a closed hyperbolic surface of genus $g \geq 2$. Then,

$$\left|\operatorname{Isom}^+(X)\right| \leq 84(g-1)$$

There is also a related bound on orders of isometries, due to Wiman: if X is of genus $g \geq 2$, every element of $\operatorname{Isom}^+(X)$ has order at most $4g+2$. A proof of this bound can be given along similar lines as the one above, but requires some more careful casework.

We now address the question of realizing these bounds. Wiman's bound is realised for all $g \geq 2$, by considering rotation of a hyperbolic $(4g+2)-$gon by "one click". Hurwitz's bound is much more interesting; it is realised for infinitely many g and *not* realised for infinitely many g. There is in fact a result of Larsen which roughly says that the frequency of g for which the bound is attained is the same as the frequency of perfect cubes in the integers.

The following result shows that the converse to the upper bound of $84(g-1)$ is also true.

Theorem 6

Let G be a finite group. Then, G can be realised as a subgroup of $\operatorname{Mod} S_g$ for some $g \geq 2$ and in fact, G is a subgroup of $\operatorname{Isom}^+(X)$ for some hyperbolic surface X homeomorphic to S_g.